UNIVERSITY OF CALGARY

On the Illumination of Three Dimensional Convex Bodies with Affine Plane Symmetry

by

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Abstract

In 1960, H. Hadwiger [27] and V. Boltyanski [15] independently posed equivalent versions of the same question: is it possible to illuminate any n-dimensional convex body by 2^n light sources? The affirmative answer to this question is called the Boltyanski-Hadwiger Illumination Conjecture. It is one of the best known open problems in Discrete Geometry and derives some of this prominence from its close relationship to the highly studied art gallery problems [44] and from its equivalence to the Levi-Gohberg-Markus Covering Conjecture [14]. In the last fifty-five years, many partial results have been proved. For example, B. V. Dekster [21] proved that eight directions illuminate three-dimensional convex bodies with affine plane symmetry. The central feature of this thesis is a rigorous exposition of most cases from Dekster's proof. Three non-trivial theorems play a significant role in the proof: the John-Löwner Theorem, the Blaschke Selection Theorem and Mazur's Finite Dimensional Density Theorem. Their proofs form another important part of this work.

Acknowledgements

The process of writing a new work is long and complicated. ... It doesn't always turn out as it was conceived. If it turns out badly, let the work remain as it is - in the next ... try to avoid ... earlier mistakes. ... When I find out that a composer has eleven versions of one symphony, I think involuntarily, how many new works might he have written in that time? — Dmitri D. Shostakovich (translated by Laurel E. Fay), *Kak rozhdaetsia muzyka*

I am indebted to Professor Károly Bezdek for generously sharing his wealth of knowledge and ideas, for his patience, sage advice, and for encouraging my independence. No words can adequately thank my parents or my husband for everything they have done to help me be successful and for their constant support.

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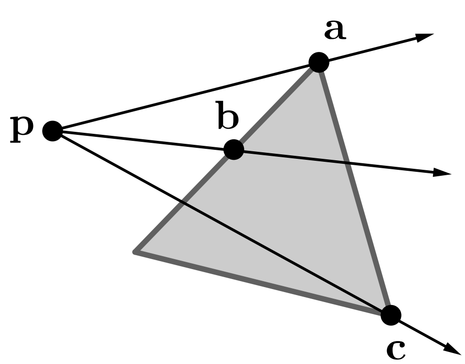
List of Symbols, Abbreviations and Nomenclature

Symbol	Definition
$\mathbb{N}$	The set of natural numbers: $\{1, 2, \ldots\}$.
$\mathbb{R}$	The set of real numbers.
$\mathbb{Z}$	The set of integers: $\{0, \pm 1, \pm 2, \ldots\}$.
$\mathbb{Z}_n$	Integers modulo n: $\{0, \ldots, n-1\}$.

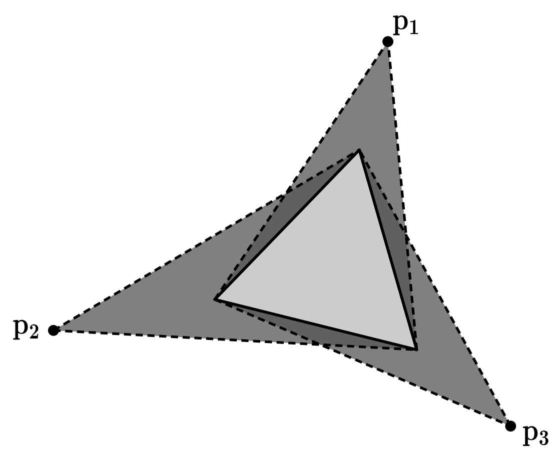
Chapter 1

Introduction

A set is *convex* if it completely contains the line segment between any two of its points. In addition to being convex, a *convex body* is a set which has interior points, includes all of its boundary points and can be completely contained in some ball. The illumination problem described by H. Hadwiger in 1960 [27] challenges geometers to find the minimum number of external light sources required to illuminate the surface of any convex body.



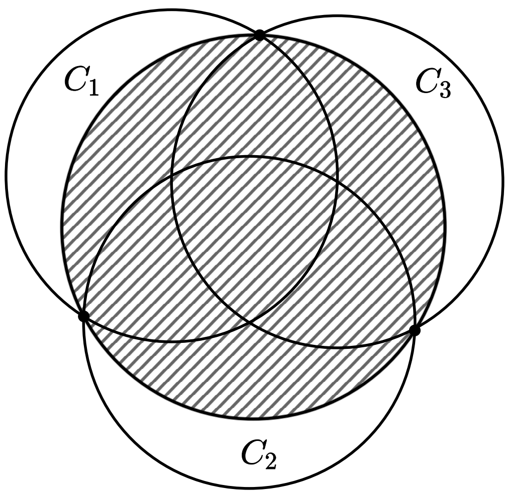
The light source **p** illuminates the point **b** on the boundary of the triangle but does not illuminate the boundary points **a** or **c**.



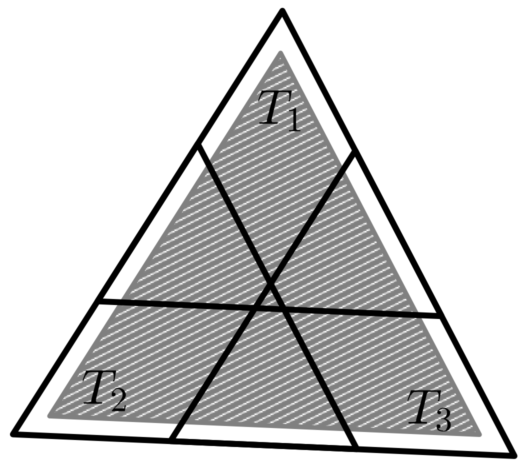
The minimum number of light sources needed to illuminate the triangle is three.

Figure 1.1

Independently, V. Boltyanski [15] asked an equivalent version of the same question in an article from 1960; instead of using external light sources for illuminating convex bodies, he proposed the use of directions. Both speculated that at most 2^n external light sources or directions were needed to illuminate the surface of an n-dimensional convex body. In the same paper [15], Boltyanski proved that the problem of illuminating convex bodies is equivalent to an earlier problem posed by I. Gohberg and A. Markus, which asks whether it is possible to cover every n-dimensional convex body by 2^n smaller copies [25]. On a historical note, F. Levi stated an equivalent version of the covering problem in 1955 and proved it in the plane [36]. Unaware of Levi's work, Gohberg and Marcus submitted their article in 1957, which also included a proof of the covering conjecture in the plane, to *Matematicheskoye Prosveshcheniye* [13]; the journal suspended publication at that time and their article was not published until 1960.



The minimum number of smaller discs required to cover the larger disc is 3.



The larger triangle can be covered by three smaller copies

Figure 1.2

The Boltyanski-Hadwiger illumination problem and the equivalent Levi-Markus-Gohberg covering problem are still open in dimensions greater than two. A solution to the illumination conjecture for 3-dimensional convex bodies was announced by Boltyanski [16]; however, the proposed proof still remains incomplete [10]. Currently, the best general upper bound on the minimum number of light sources required to illuminate a 3-dimensional convex body, in the literature, is 16 and is due to I. Papadoperakis [45].

Many results of the illumination and covering conjectures for special kinds of convex bodies have been established. For example, it is known that d-dimensional convex bodies whose boundaries consist only of smooth points can be illuminated or covered by $d+1$ light sources or smaller copies, respectively (see [36], [15], [12]). In addition, K. Bezdek proved the illumination conjecture holds for 3-dimensional convex polyhedra with affine symmetry [11], M. Lassak proved that the illumination conjecture holds for centrally symmetric 3-dimensional convex bodies [32] and B.V. Dekster proved the illumination conjecture holds for 3-dimensional convex bodies with affine plane symmetry [21]. For a more comprehensive account of the major results known about the illumination and covering conjectures and their applications, the interested reader can consult the surveys in [10], [18] and [58].

The central focus of this thesis is to provide a rigorous account of B.V. Dekster's partial result [21]. Chapter 2 states definitions and basic theorems, which are required in Chapters 3 and 4. The proof of the illumination conjecture for 3-dimensional convex bodies with affine plane symmetry relies on three non-trivial theorems: the Blaschke Selection Theorem, Mazur's Finite Dimensional Density Theorem and the John-Löwner Theorem. These three theorems are proved in Chapter 3. Finally, Chapter 4 gives a rigorous exposition of Dekster's proof [21].

Chapter 2

Preliminaries

2.1 Euclidean n-Space

Let n be some positive integer strictly greater than 1. The set $\mathbb{R}^n$ is defined as all ordered n-tuples of real numbers; namely, $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R},\ 1 \leqslant i \leqslant n\}$. An element $(x_1, x_2, \ldots, x_n)$ of $\mathbb{R}^n$ is denoted by $\mathbf{x}$ and called a *vector* or *point*, interchangeably. Vectors consist of *coordinates*. Specifically, the real numbers x_i, for all $1 \leqslant i \leqslant n$, are the coordinates of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Addition between any two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is defined as follows:

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n).$$

Likewise, multiplication of any vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{E}^n$ by a scalar $\lambda \in \mathbb{R}$ is defined by

$$\lambda\mathbf{x} = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n).$$

Geometrically, two vectors in $\mathbb{R}^n$ are said to be *parallel* if one can be written as a scalar multiple of the other; in other words, the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are parallel if there exists a real number λ such that $\mathbf{x} = \lambda\mathbf{y}$. The vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are said to have the *same direction* if there exists a real number $\lambda \geqslant 0$ such that $\mathbf{x} = \lambda\mathbf{y}$. The vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are said to have *opposite directions* if there exists a real number $\lambda < 0$ such that $\mathbf{x} = \lambda\mathbf{y}$.

With the operations of vector addition and scalar multiplication defined above, $\mathbb{R}^n$ determines a vector space over the field of real numbers, $\mathbb{R}$. The additive identity in $\mathbb{R}^n$, known as the *origin* or the *zero vector*, is denoted by $\mathbf{o}$.

The operations of vector addition and scalar multiplication can be extended to sets, as follows. Given any two sets A and B in $\mathbb{R}^n$, the *Minkowski sum* between these two sets is defined by

$$A + B = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A \text{ and } \mathbf{b} \in B\}.$$

The Minkowski sum between a set A and some singleton set $\{\mathbf{x}\}$ is called a *translate* of A by $\mathbf{x}$; it is commonly written as $\mathbf{x} + A$. The Minkowski sum, $A + B$, may also be expressed as the union of translates

$$\bigcup_{\mathbf{a}\in A}(\mathbf{a} + B) = \bigcup_{\mathbf{b}\in B}(A + \mathbf{b}).$$

Proposition 2.1.1. *Let A, B and C be subsets of $\mathbb{R}^n$. If $A \subseteq B$, then $A + C \subseteq B + C$.*

Proof. Suppose $A \subseteq B$. Let $\mathbf{x} \in A + C$ be arbitrarily chosen. Then, there exists $\mathbf{a} \in A$ and $\mathbf{c} \in C$ such that $\mathbf{x} = \mathbf{a} + \mathbf{c}$. However, $\mathbf{a} \in A \subseteq B$. Therefore, $\mathbf{x} = \mathbf{a} + \mathbf{c} \in B + C$. ■

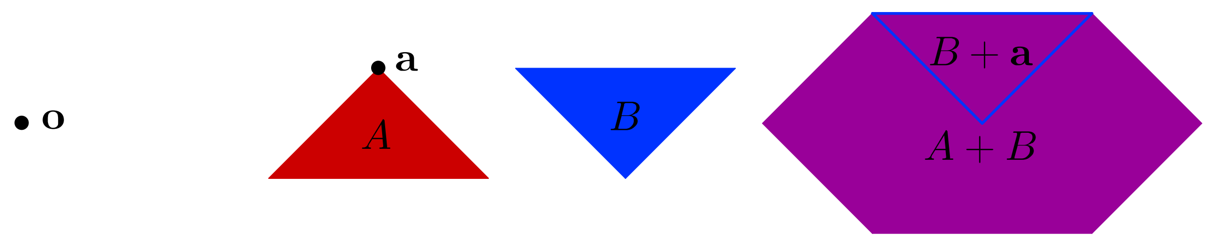


Figure 2.1: Minkowski Sum of Two 2-simplices

For any scalar $\lambda \in \mathbb{R}$ and any set A in $\mathbb{R}^n$, the set $\lambda A = \{\lambda\mathbf{a} \mid \mathbf{a} \in A\}$ is called a *scalar multiple* of A. The set A is said to be *homothetic* to the set B if there exists some real number $\lambda \neq 0$ and some vector $\mathbf{x} \in \mathbb{R}^n$ such that $A = \lambda B + \mathbf{x}$.

There is another operation between sets, known as the *Cartesian product*. For any two sets A and B in $\mathbb{R}^n$, the Cartesian product of these two sets is defined and denoted by

$$A \times B = \{(\mathbf{a}, \mathbf{b}) \mid \mathbf{a} \in A \text{ and } \mathbf{b} \in B\},$$

where $(\mathbf{a}, \mathbf{b})$ is an ordered $2n$-tuple and $A \times B \in \mathbb{R}^{n \times n} = \mathbb{R}^{n^2}$.

In addition to the operations of addition and scalar multiplication, there is another operation between vectors in $\mathbb{R}^n$ known as the *inner product*. Given two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, the inner product is a map which sends the ordered pair $(\mathbf{x}, \mathbf{y})$ in $\mathbb{R}^n \times \mathbb{R}^n$ to the real number $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$. Notice that $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ and that $\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \lambda \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$, for any real number λ.

The inner product gives rise to the concept of the length or norm of a vector in $\mathbb{R}^n$, sometimes called the *Euclidean norm*. The Euclidean norm is defined to be a map which sends a vector $\mathbf{x}$ in $\mathbb{R}^n$ to the real number $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$. Notice that that $\|\mathbf{x}\| > 0$ for all $\mathbf{x} \neq \mathbf{o}$ and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{o}$. Also, notice that $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|$, for any real number λ.

Vectors in $\mathbb{R}^n$ of length one are called *unit vectors*. The set of all unit vectors in $\mathbb{E}^n$ is the $n-1$-dimensional sphere:

$$\mathcal{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| = 1\}.$$

The *Euclidean distance* between any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ is found by taking the Euclidean norm of the vectors $\mathbf{x} - \mathbf{y}$ or $\mathbf{y} - \mathbf{x}$: namely, it is the real number found by calculating $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{y} - \mathbf{x}\|$.

Stated below are two well-known inequalities: the Cauchy-Schwarz inequality and the triangle inequality. Their proofs can be found on p. 3 of [9]. The so-called reverse triangle inequality is also stated below. Its proof can be found on p. 584 of [24].

Theorem 2.1.2. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *be arbitrarily chosen. Then,*

(i) (Cauchy-Schwarz Inequality) $|\langle \mathbf{x}, \mathbf{y} \rangle| \leqslant \|\mathbf{x}\| \|\mathbf{y}\|$

(ii) (Triangle Inequality) $\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$

(iii) (Reverse Triangle Inequality) $|\|\mathbf{x}\| - \|\mathbf{y}\|| \leqslant \|\mathbf{x} - \mathbf{y}\|$

Note that equality holds for the Cauchy-Schwarz inequality if and only if either $\mathbf{x} = \lambda\mathbf{y}$ for some real number λ or $\mathbf{y} = \mu\mathbf{x}$ for some real number μ. Equality occurs in the triangle inequality if and only if either $\mathbf{x} = \lambda\mathbf{y}$ for some real number $\lambda \geqslant 0$.

The *angle* between any non-zero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ is the real number θ, which satisfies

$$\cos(\theta) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\|\mathbf{y}\|}$$

in the interval $0 \leqslant \theta \leqslant \pi$. The angle θ is uniquely determined in this interval. Notice that the angle between the vectors $\lambda\mathbf{x}$ and $\mu\mathbf{y}$, for any real numbers $\lambda, \mu > 0$, is equivalent to the angle between $\mathbf{x}$ and $\mathbf{y}$.

The vector space $\mathbb{R}^n$ together with the Euclidean distance is a metric space called the *n-dimensional Euclidean space* and is denoted by $\mathbb{E}^n$.

2.2 Linear and Affine sets

A set S in $\mathbb{E}^n$ is called a *linear subspace* of $\mathbb{E}^n$ if for each pair of vectors $\mathbf{x}, \mathbf{y} \in S$ and for any scalar $\lambda \in \mathbb{R}$, $\lambda\mathbf{x} + \mathbf{y} \in S$. Similarly, a set A in $\mathbb{E}^n$ is said to be *affine* if for each pair of vectors $\mathbf{x}, \mathbf{y} \in A$ and for $\lambda \in \mathbb{R}$, $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in A$. In other words, a set is affine if for any two vectors in the set, the entire line through the vectors is contained by the set. Affine sets and linear subspaces of $\mathbb{E}^n$ relate to each other in the following way; any affine set in $\mathbb{E}^n$ containing the origin is a linear subspace of $\mathbb{E}^n$. Two affine sets A and B in $\mathbb{E}^n$ are *parallel* if one is a translate of the other.

A *linear combination* of the vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}$ in $\mathbb{E}^n$ is $\lambda_1\mathbf{x_1} + \lambda_2\mathbf{x_2} + \ldots + \lambda_m\mathbf{x_m}$ for any real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$. If the scalars in the above linear combination satisfy the further condition that $\lambda_1 + \lambda_2 + \ldots + \lambda_m = 1$, then $\lambda_1\mathbf{x_1} + \lambda_2\mathbf{x_2} + \ldots + \lambda_m\mathbf{x_m}$ is called an *affine combination* of the vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}$ in $\mathbb{E}^n$. Given a set A in $\mathbb{E}^n$, the set of all affine combinations of the vectors of A is the *affine hull* of A, denoted by $\text{aff}(A)$. The affine hull, aff(A), can also be described as the intersection of all affine sets in $\mathbb{E}^n$ containing A. It should be noted that the intersection of an arbitrary family of affine sets is an affine set. Also, recall the following. Given a set S in $\mathbb{E}^n$, S is said to be *linearly dependent* if there exist distinct vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}$ in S and scalars $\lambda_1, \lambda_2, \ldots, \lambda_m$, not all zero, such that $\lambda_1\mathbf{x_1} + \lambda_2\mathbf{x_2} + \ldots + \lambda_m\mathbf{x_m} = 0$. A set that is not linearly dependent is said to be linearly independent.

2.2.1 Dimension

Let S be some linear subspace of $\mathbb{E}^n$. If the vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}$ in S are linearly independent and if S can be written as the set of all linear combinations of these vectors, then $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}\}$ is called the *linear basis* of S. The number of vectors in the linear basis is the *dimension* of the linear subspace S, denoted dim(S). The dimension of an affine set A is the dimension of the linear subspace parallel to it. The dimension of a set B in $\mathbb{E}^n$ is the dimension of the smallest affine set containing B, i.e. the aff(B).

2.2.2 Examples of Linear and Affine Sets

Given the terminology developed above, examples of linear and affine sets can now be meaningfully provided.

The empty set $\varnothing$ is a -1-dimensional affine set, singleton sets are 0-dimensional affine sets, lines are 1-dimensional affine sets, planes are 2-dimensional affine sets and *hyperplanes* are $n-1$-dimensional affine sets in $\mathbb{E}^n$.

The definition of parallel affine set given above does not obviously describe the behaviour of two parallel lines in a plane but the following theorem does; the proof of the forwards direction can be found on page 16 of [60] and the backwards direction follows directly from the definition of affine parallel sets.

Theorem 2.2.2.1. *Two distinct lines ℓ_1 and ℓ_2, in some plane P of $\mathbb{E}^n$, do not intersect if and only if ℓ_1 and ℓ_2 are parallel.*

Likewise, the singleton set $\{0\}$ is the 0-dimensional linear set, lines passing through the origin are 1-dimensional linear sets, planes containing the origin are 2-dimensional linear sets and hyperplanes containing the origin are $n-1$-dimensional linear sets.

2.2.3 Hyperplanes, Halfspaces and Slabs

A hyperplane H in $\mathbb{E}^n$ is defined to be $\{\mathbf{x} \in \mathbb{E}^n \mid \langle \mathbf{x}, \mathbf{u} \rangle = \lambda,\ \mathbf{u} \in \mathcal{S}^{n-1},\ \lambda \in \mathbb{R}\}$. Note that the unit vector $\mathbf{u}$ is normal to H. Two hyperplanes are parallel if and only if their unit normal vectors are a scalar multiples.

The hyperplane H divides $\mathbb{E}^n$ into two *half-spaces*. Namely, the set of vectors lying strictly to one side or the other of H are expressed as

$$H^+ = \{\mathbf{x} \in \mathbb{E}^n \mid \langle \mathbf{x}, \mathbf{u} \rangle > \lambda,\ \mathbf{u} \in \mathcal{S}^{n-1},\ \lambda \in \mathbb{R}\}$$

and

$$H^- = \{\mathbf{x} \in \mathbb{E}^n \mid \langle \mathbf{x}, \mathbf{u} \rangle < \lambda,\ \mathbf{u} \in \mathcal{S}^{n-1},\ \lambda \in \mathbb{R}\}.$$

The half-spaces H^+ and H^- are *open half-spaces* determined by H. The *closed half-spaces* determined by H are the set of vectors lying on and to one side or the other of H. They are denoted by and defined as $\overline{H}^+ = \{\mathbf{x} \in \mathbb{E}^n \mid \langle \mathbf{x}, \mathbf{u} \rangle \geqslant \lambda,\ \mathbf{u} \in \mathcal{S}^{n-1},\ \lambda \in \mathbb{R}\}$ and $\overline{H}^- = \{\mathbf{x} \in \mathbb{E}^n \mid \langle \mathbf{x}, \mathbf{u} \rangle \leqslant \lambda,\ \mathbf{u} \in \mathcal{S}^{n-1},\ \lambda \in \mathbb{R}\}$.

A *slab* in $\mathbb{E}^n$ is the closed connected region bounded by two distinct parallel hyperplanes. Specifically, the slab between the hyperplanes $H_1 = \{\mathbf{x} \in \mathbb{E}^n \mid \langle \mathbf{x}, \mathbf{u} \rangle = \lambda_1,\ \mathbf{u} \in \mathcal{S}^{n-1}\}$ and $H_2 = \{\mathbf{x} \in \mathbb{E}^n \mid \langle \mathbf{x}, \mathbf{u} \rangle = \lambda_2,\ \mathbf{u} \in \mathcal{S}^{n-1}\}$, for $\lambda_1 < \lambda_2$, is expressed by

$$\text{slab}[H_1, H_2] = \{\mathbf{x} \in \mathbb{E}^n \mid \lambda_1 \leqslant \langle \mathbf{x}, \mathbf{u} \rangle \leqslant \lambda_2,\ \mathbf{u} \in \mathcal{S}^{n-1}\}.$$

2.3 Matrices

A rectangular array of real numbers with m rows and n columns is called an $m \times n$ *matrix* and belongs to $\mathbb{E}^{m\times n}$. The (i, j)-*entry* of a real-valued $m \times n$ matrix A is is the number in the i-th row and j-th column of A and denoted by a_{ij}. Two matrices A and B are *equal* if they have the same number of rows and columns, and if $a_{ij} = b_{ij}$ for all possible values of i and j.

Let A and B be two matrices with the same number of rows and columns. The sum $A + B$ is the matrix consisting of the entries $a_{ij} + b_{ij}$ for each i and j. If λ be some real number and A some matrix, then the matrix λA consists of the entries λa_{ij} for each i and j. Let A be an $m \times n$ matrix. The $n \times m$ matrix whose entries are a_{ji} for any $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant m$ is called the *transpose* of A and is denoted by A^T. A matrix with the same number of rows as columns is called *square*. If A is square matrix with the property that $A = A^T$, then A is called *symmetric*. A square matrix A is called *diagonal* if $a_{ij} = 0$ for all $i \neq j$. Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. Denote the i-th row of A by $\mathbf{a}_i = \begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{pmatrix}$ and denote the j-column of B by $\mathbf{b}'_j = \begin{pmatrix} b_{j1} & b_{j2} & \dots & b_{jn} \end{pmatrix}$. The *product* AB is the $m \times k$ matrix whose (i, j)-entry is

$$\left\langle \mathbf{a}_i, \mathbf{b}'_j \right\rangle = a_{i1}b'_{j1} + a_{i2}b'_{j2} + \ldots + a_{in}b'_{jn} = \mathbf{a}_i^T \mathbf{b}'_j.$$

Note that, in general, $AB \neq BA$. The $n \times n$ diagonal matrix whose (i, i)-entries are equal to 1, for each $1 \leqslant i \leqslant n$, is called the *identity matrix* and is denoted by I_n. Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix B such that $AB = I_n$ and $BA = I_n$, then A is said to be *invertible* and B is called the *inverse* of A; B is often written as A^{-1}.

Below is a collection of several helpful properties of the transpose, matrix multiplication and matrix inverses. Their proofs can be found on pages 32, 47, 56, 57 and 45 of [43] and page 468 of [54], respectively.

Properties 2.3.1.

(i) *Let* A *be an* $m \times n$ *matrix. Then,* $\left(A^T\right)^T = A$*;*

(ii) *Let* A *be an* $m \times n$ *matrix. Then* $I_m A = A = A I_n$*;*

(iii) *Let the matrices* B *and* C *have the same number of rows and columns. Then,* $A\,(B \pm C) = AB \pm AC$ *and* $(B \pm C)\,A = BA \pm CA$*, if the matrix* A *is sized so that the products are defined;*

(iv) *Let* A *and* B *be matrices whose product is defined and let* $\lambda \in \mathbb{R}$*. Then,* $\lambda\,(AB) = (\lambda A)\,B = A\,(\lambda B)$*.*

(v) *Let* A *and* B *be two compatible matrices. Then,* $(AB)^T = B^T A^T$*;*

(vi) *If* A *is an invertible* $n \times n$ *matrix, then* $(A^{-1})^{-1} = A$*;*

(vii) *Let* A *and* B *be* $n \times n$ *invertible matrices. Then,* $(AB)^{-1} = B^{-1}A^{-1}$*;*

(viii) *Let* A *be an invertible* $n \times n$ *matrix and let* λ *be a non-zero scalar. Then,* $(\lambda A)^{-1} = \dfrac{1}{\lambda} A^{-1}$*;*

(ix) *Let* A *be an invertible matrix. Then,* $\left(A^T\right)^{-1} = (A^{-1})^T$*.*

(x) *Let* A_1 *and* A_2 *be* $n \times n$ *matrices, let* B_1 *and* B_2 *be* $m \times m$ *matrices, and let*

$$C_1 = \begin{bmatrix} A_1 & X_1 \\ 0 & B_1 \end{bmatrix}$$

and

$$C_2 = \begin{bmatrix} A_2 & X_2 \\ 0 & B_2 \end{bmatrix}$$

be block matrices where 0 *denotes an* $(m-n) \times (m-n)$ *matrix whose entries are all zeros. Then,*

$$C_1 C_2 = \begin{bmatrix} A_1 A_2 & A_1 X_2 + X_1 B_2 \\ 0 & B_1 B_2 \end{bmatrix}.$$

(xi) Let A and B be square matrices and let

$$C = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$$

be a block matrix where 0 is a matrix whose entries are all zeros. Then, C is invertible if and only if A and B are invertible and

$$C^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}XB^{-1} \\ 0 & B^{-1} \end{bmatrix}.$$

2.4 Linear and Affine Transformations

A *function* $T : \mathbb{E}^n \to \mathbb{E}^m$ is a rule that assigns to every vector $\mathbf{x} \in \mathbb{E}^n$ a uniquely determined vector $T(\mathbf{x})$ in $\mathbb{E}^m$. Below, a basic but useful fact about functions is stated; see page A58 of [22] for its proof.

Proposition 2.4.1. *Let $T : X \to Y$ be a function and let $A \subseteq B \subseteq X$. Then, $T(A) \subseteq T(B)$.*

Given the functions $S, T : \mathbb{E}^n \to \mathbb{E}^m$ and $\lambda \in \mathbb{R}$, define the sum $S + T$, the scalar product λT and composition $S \circ T$ by

$$(S + T)(\mathbf{x}) = S(\mathbf{x}) + T(\mathbf{x}), \quad (\lambda T)(\mathbf{x}) = \lambda T(\mathbf{x})$$

and

$$(S \circ T)(\mathbf{x}) = S(T(\mathbf{x})),$$

for all $\mathbf{x} \in \mathbb{E}^n$. The function $i_{\mathbb{E}^n} : \mathbb{E}^n \to \mathbb{E}^n$ defined by $i_{\mathbb{E}^n}(\mathbf{x}) = \mathbf{x}$, for all $\mathbf{x} \in \mathbb{E}^n$ is called the *identity function.* A function $T : \mathbb{E}^n \to \mathbb{E}^m$ is said to be *invertible* if there exists a functions $T', T^\star : \mathbb{E}^m \to \mathbb{E}^n$ such that $T \circ T' = i_{\mathbb{E}^m}$ and $T^\star \circ T = i_{\mathbb{E}^n}$. The proposition below specifies the relationship between the functions T' and $T^\star$; its proof can be found on page 22 of [57].

Proposition 2.4.2. *Let $T : \mathbb{E}^n \to \mathbb{E}^m$ be a function such that $T \circ T' = i_{\mathbb{E}^m}$ and $T^\star \circ T = i_{\mathbb{E}^n}$ for some functions $T', T^\star : \mathbb{E}^m \to \mathbb{E}^n$. Then, $T' = T^\star$ and the function T', called the inverse of T, is unique.*

Let $T : \mathbb{E}^n \to \mathbb{E}^m$ be a function with the property that for each element $\mathbf{y} \in \mathbb{E}^m$ there exists a unique element $\mathbf{x} \in \mathbb{E}^n$ such that $y = T(\mathbf{x})$. Then, the function T is called a *bijection*. The following lemma connects the concepts of invertible functions to bijections; its proof can be found on page 128 of [31].

Lemma 2.4.3. *A function $T : \mathbb{E}^n \to \mathbb{E}^m$ is a bijection if and only if T is invertible.*

If a function $T : \mathbb{E}^n \to \mathbb{E}^m$ satisfies the conditions

$$T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) \quad \text{and} \quad T(\lambda \mathbf{x}) = \lambda T(\mathbf{x})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{E}^n$ and $\lambda \in \mathbb{R}$, then T is called a *linear transformation*. The following theorem relates linear transformations to matrices. Its proof can be found on pages 75 and 76 of [43].

Theorem 2.4.4. *Let $T : \mathbb{E}^n \to \mathbb{E}^m$ be a transformation. If T is a linear transformation, then T is induced by a unique matrix, $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n) \end{bmatrix}$; namely, $T(\mathbf{x}) = A\mathbf{x}$ for all $x \in \mathbb{E}^n$. If T is induced by an $m \times n$ matrix A, then T is a linear transformation.*

The theorem below provides a criterion for determining whether a linear function is invertible. It is proved on page 79 of [43].

Theorem 2.4.5. *Let A be an $n \times n$ matrix which induces the linear transformation $T : \mathbb{E}^n \to \mathbb{E}^n$. Then, T is invertible if and only if the matrix A is invertible.*

An important fact about the inverse of a linear transformation is exhibited in the next lemma, which is proved on page 128 of [31].

Lemma 2.4.6. *If $T : \mathbb{E}^n \to \mathbb{E}^n$ is an invertible linear transformation, then its inverse T^{-1} is also a linear transformation.*

In comparison, a transformation is called *affine* if it satisfies the property

$$T(\lambda \mathbf{x} + \mu \mathbf{y}) = \lambda T(\mathbf{x}) + \mu T(\mathbf{y}),$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{E}^n$ and any $\lambda, \mu \in \mathbb{R}$ such that $\lambda + \mu = 1$. Notice that every linear transformation is an affine transformation. However, the converse is not true: an affine transformation $T : \mathbb{E}^n \to \mathbb{E}^m$ is linear if and only if $T(\mathbf{o}) = \mathbf{o}$ (see Theorem 1.5.1 in [60]). The following theorem allows us to more precisely understand the relationship between linear and affine transformations. The first half of its proof can be found on p. 23 of [60]. The latter half is not difficult to prove and it can be done with the help of *(iii)* and *(iv)* from Properties 2.3.1.

Theorem 2.4.7. *Let $T : \mathbb{E}^n \to \mathbb{E}^m$ be a transformation. If T is an affine transformation, then T is induced by the unique $m \times n$ matrix,*

$$A = \Big[T(\mathbf{e}_1) - T(\mathbf{o}) \quad T(\mathbf{e}_2) - T(\mathbf{o}) \quad \dots \quad T(\mathbf{e}_n) - T(\mathbf{o}) \Big]$$

and the translate $\mathbf{b} = T(\mathbf{o})$; namely, $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for any $\mathbf{x} \in \mathbb{E}^n$. If $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for all $\mathbf{x} \in \mathbb{E}^n$, then T is an affine transformation.

Below are a collection of properties of the affine transformation, which geometrically describe its action. Proofs of these properties can be found on pages 23 of [60], page 24 of [59], page 379 of [60], page 92 of [17], on pages 4 and 5 of [52], respectively.

Properties 2.4.8. *Let $T : \mathbb{E}^n \to \mathbb{E}^m$ be an affine transformation.*

(i) *For any $A \subseteq \mathbb{E}^n$, $T\big(\mathrm{aff}(A)\big) = \mathrm{aff}\big(T(A)\big)$. Hence, $T(A)$ is an affine set if A is an affine set.*

(ii) *If $A \subseteq \mathbb{E}^n$ is a convex set, then $T(A)$ is convex.*

(iii) *Let $A, B \subseteq \mathbb{E}^n$ be parallel affine sets; namely, $A = B + \mathbf{t}$ for some $\mathbf{t} \in \mathbb{E}^n$. Then, the flats $T(A)$ and $T(B)$ are parallel in $\mathbb{E}^m$; i.e., there exists $\mathbf{t}' \in \mathbb{E}^m$ such that $T(A) = T(B) + \mathbf{t}'$.*

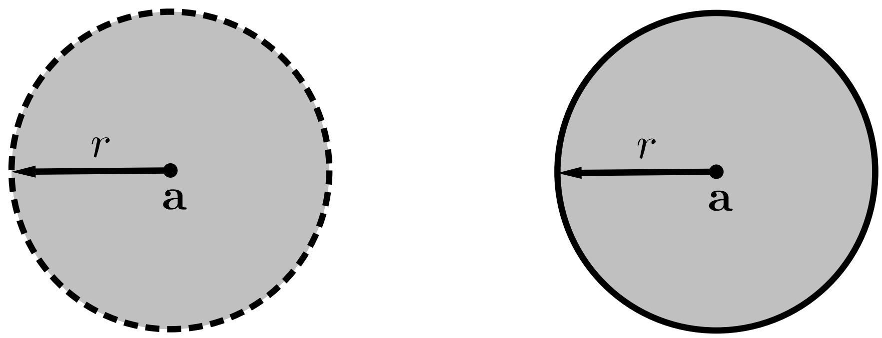
(iv) *Affine transformations preserve ratios of lengths along parallel lines.*

(v) *If $A \subseteq \mathbb{E}^n$ is a convex set, then T maps the extreme points of A onto the extreme points of $T(A)$.*

(vi) *For any $A \subseteq \mathbb{E}^n$, $T\big(\mathrm{conv}(A)\big) = \mathrm{conv}\big(T(A)\big)$.*

2.5 Open and Closed Sets

The *open ball* with radius r and centre $\mathbf{a} \in \mathbb{E}^n$ is the set of all vectors in $\mathbb{E}^n$ whose distance from the vector $\mathbf{a}$ is strictly less than r and is denoted by $\mathrm{B}(\mathbf{a}, r)$; i.e., $\mathrm{B}(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{E}^n \mid \|\mathbf{a} - \mathbf{x}\| < r\}$. Similarly, the *closed ball* with radius r and centre $\mathbf{a}$ is denoted and defined by $\mathrm{B}[\mathbf{a}, r] = \{\mathbf{x} \in \mathbb{E}^n \mid \|\mathbf{a} - \mathbf{x}\| \leqslant r\}$.



Open disc, $\mathrm{B}(\mathbf{a}, r)$, in $\mathbb{E}^2$ Closed disc, $\mathrm{B}[\mathbf{a}, r]$, in $\mathbb{E}^2$

Figure 2.2

An element $\mathbf{x}$ of a set S in $\mathbb{E}^n$ is called an *interior point* of S if there exists a real number $r > 0$ such that an open ball with radius r whose centre is $\mathbf{x}$ is contained in S; i.e., $\mathrm{B}(\mathbf{x}, r) \subseteq S$. The set of all interior points of S is called the *interior* of S and is denoted by $\mathrm{int}(S)$. Similarly, the *relative interior* of a set S is the collection of all elements $\mathbf{x} \in S$ such that $B(\mathbf{x}, r) \cap \mathrm{aff}(S) \subseteq S$, for some real number $r > 0$. In other words, the relative interior of a set S is its interior relative to its affine hull. Denote the relative interior of a set S by $\mathrm{relint}(S)$. Note that the relative interior of any affine set in $\mathbb{E}^n$ is itself and if $\mathrm{int}(S) \neq \emptyset$, then $\mathrm{relint}(S) = \mathrm{int}(S)$ (see page 37 of [60]).

A set S is said to be *open* if each of its elements is an interior point of S. A set S is said to be *closed* if its complement $\mathbb{E}^n \backslash S$ is open. The intersection of all closed sets containing the set S we call the *closure* of S and denote it by $\mathrm{cl}(S)$.

Below, is a collection of properties and examples of open and closed sets; proofs of all but the last statement can be found on page 94 of [42], page 35 of [50], page 22 of [1], page 33 of [60], page 328 of [46], page 36 of [60] and page 99 of [42], respectively. The last statement follows from the preceding statement and Corollary 2.10.8.

Theorem 2.5.1.

(i) *Arbitrary intersections of closed sets are closed and arbitrary unions of open sets are open.*

(ii) *The closure of any set is closed and the interior of any set is open.*

(iii) *Let* $A \subseteq X$. *If* A *is closed, then* $\mathrm{cl}(A) = A$. *If* A *is open, then* $A = \mathrm{int}(A)$.

(iv) *Open balls in* $\mathbb{E}^n$ *are open.*

(v) $\mathcal{S}^{n-1} = \{\mathbf{z} \in \mathbb{E}^n \mid \|\mathbf{z}\| = 1\}$ *is closed in* $\mathbb{E}^n$.

(vi) *Affine sets in* $\mathbb{E}^n$ *are closed.*

(vii) *Finite point sets in* $\mathbb{E}^n$ *are closed.*

(viii) *Closed line segments in* $\mathbb{E}^n$ *are closed.*

A *topology* on a set X is a collection, $\mathcal{T}$, of subsets of X such that $\varnothing$ and X belong to $\mathcal{T}$, the union of the elements from any sub-collection of $\mathcal{T}$ is also in $\mathcal{T}$, and the intersection of any finite sub-collection of $\mathcal{T}$ also belongs to $\mathcal{T}$. The set X together with a topology $\mathcal{T}$ is called a *topological space*. Euclidean n-space is an example of a topological space (see page 142 of [2]). A proof of the following statement can be found on page 94 of [42].

Theorem 2.5.2. *Let* Y *be a subset of the topological space* $(X, \mathcal{T})$. *Then, a set* A *is closed in* Y *with the subspace topology,* $\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$, *if and only if* A *can be written as the intersection of a closed set of* $(X, \mathcal{T})$ *with* Y.

A proof of the following statement is on page 96 of [42].

Theorem 2.5.3. *Let A be a subset of some topological space X. Then, $x \in \text{cl}(A)$ if and only if every open set U containing x intersects A.*

The following theorem shows the relationship between interiors of sets and closures of sets.

Theorem 2.5.4. *Let A be a subset of some topological space X. Then,*

$$\text{cl}(X \backslash A) = X \backslash \text{int}(A).$$

Proof. First, it will be shown that $\text{cl}(X\backslash A) \subseteq X\backslash \text{int}(A)$. Recall that $\text{int}(A)$ is an open set. It follows by definition that $X\backslash\text{int}(A)$ is closed. Let $x \in X\backslash A$ be arbitrarily chosen. Then, $x \in X$ and $x \notin A$. It follows that $x \notin \text{int}(A)$ since $\text{int}(A) \subseteq A$. This implies that $x \in X\backslash\text{int}(A)$. Therefore, $X\backslash A \subseteq X\backslash\text{int}(A)$. However, by definition $\text{cl}(X\backslash A)$ is the smallest closed set containing $X\backslash A$. Thus, $\text{cl}(X\backslash A) \subseteq X\backslash\text{int}(A)$.

Now, it will be shown that $X\backslash\text{int}(A) \subseteq \text{cl}(X\backslash A)$. Let $x \in X\backslash\text{int}(A)$ be arbitrarily chosen. Then, $x \in X$ and $x \notin \text{int}(A)$. This implies that for every open set U containing x in X, $U \nsubseteq A$. This means that $U \cap (X\backslash A) \neq \varnothing$, for every open set U containing x. By Theorem 2.5.3, it follows that $x \in \text{cl}(X\backslash A)$. Hence, $X\backslash\text{int}(A) \subseteq \text{cl}(X\backslash A)$. ■

A proof of the elementary but useful fact below can be found on page 1 of [29].

Lemma 2.5.5. *Let $A \subset B$ in some topological space X. Then,* $\text{int}(A) \subseteq \text{int}(B)$.

The next result follows from Lemma 2.5.5 and Theorem 2.5.4.

Corollary 2.5.6. *Let $A \subset B$ in some topological space X. Then,* $\text{cl}(A) \subseteq \text{cl}(B)$.

The *boundary* of a set S, which we denote by $\text{bd}(S)$, is the intersection of the closure of S with the closure of its complement $\mathbb{E}^n\backslash S$, i.e. $\text{bd}(S) = \text{cl}(S) \cap \text{cl}(\mathbb{E}^n\backslash S)$. The *relative boundary* of a set S, $\text{relbd}(S)$, are all the elements which lie in the closure of S but which do not lie in the relative interior of S. It follows directly from this definition that $\text{relbd}(S) \cap \text{relint}(S) = \varnothing$ and that $\text{relbd}(S) \cup \text{relint}(S) = \text{cl}(S)$. Also, note that if $\text{aff}(S) = \mathbb{E}^n$, $\text{relbd}(S) = \text{bd}(S)$. Furthermore, the relative boundary of any affine set in $\mathbb{E}^n$ is empty.

Theorem 2.5.7. *For any set S,* $\mathrm{cl}(S) = \mathrm{int}(S) \cup \mathrm{bd}(S)$ *and* $\mathrm{int}(S) \cap \mathrm{bd}(S) = \varnothing$. *Consequently,* $\mathrm{bd}(S) = \mathrm{cl}(S) \backslash \mathrm{int}(K)$.

2.6 Sequences

A sequence $\{\mathbf{s}_k\}_{k\in\mathbb{N}}$ in $\mathbb{E}^n$ *converges* to the vector $\mathbf{s}$ if for each real $\varepsilon > 0$ there exists an integer N such that for all $k \in \mathbb{N}$ where $k > N$, $\|\mathbf{s_k} - \mathbf{s}\| < \varepsilon$. Given the sequence $\{\mathbf{s}_k\}_{k\in\mathbb{N}}$ in $\mathbb{E}^n$ and a sequence $k_{i i\in\mathbb{N}}$ of positive integers, such that $k_1 < k_2 < k_3 < \ldots$, the sequence $\{\mathbf{s}_{k_i}\}_{i\in\mathbb{N}}$ is called a *subsequence* of $\{\mathbf{s}_k\}_{k\in\mathbb{N}}$.

Theorem 2.6.1. *A sequence $\{\mathbf{s}_k\}_{k\in\mathbb{N}}$ converges to $\mathbf{s}$ if and only if every subsequence of $\{\mathbf{s}_k\}_{k\in\mathbb{N}}$ converges to $\mathbf{s}$.*

A proof of the following theorem can be found on page 99 of [42].

Theorem 2.6.2. *A sequence of points of $\mathbb{E}^n$ converges to at most one point of $\mathbb{E}^n$.*

See Lemma 21.2 on page 130 of [42] for a proof of the theorem below.

Theorem 2.6.3. *Let X be a topological space and let $S \subseteq X$. If there exists a sequence of points of S which converges to x, then $x \in \mathrm{cl}(S)$. Moreover, if there exists a metric $\hat{d}$ on X, then the converse also holds.*

The next theorem describes a useful property for sequences of real numbers; a proof of the statement can be found on page 168 of [34].

Theorem 2.6.4. *Let $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ be two convergent sequences of real numbers, which converge to $x \in \mathbb{R}$ and $y \in \mathbb{R}$ respectively. If $x_k \leqslant y_k$ for all $k \in \mathbb{N}$, then $x \leqslant y$.*

2.7 Bounded Sets

A set S in $\mathbb{E}^n$ is said to be *bounded* if there exists some real number $M \geqslant 0$ such that for every pair of vectors $\mathbf{x_1}, \mathbf{x_2} \in S$,

$$\|\mathbf{x_1} - \mathbf{x_2}\| \leqslant M.$$

An extremely useful property of bounded sequences in $\mathbb{E}^n$, known as the Bolzano–Weierstrass theorem, is stated below; a proof of this statement can be found on page 39 of [60].

Theorem 2.7.1. *Every bounded sequence of points in $\mathbb{E}^n$ contains a convergent subsequence.*

One consequence of the Bolzano-Weierstrass theorem is the following statement.

Corollary 2.7.2. *Every bounded divergent sequence in $\mathbb{E}^n$, has at least two limit points.*

A well-known property for bounded sequences of real numbers that are either non-increasing or non-decreasing, known as the monotone convergence theorem, is stated below; its proof can be found on page 175 of [34].

Theorem 2.7.3. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a bounded sequence of real numbers such that either $x_k \leqslant x_{k+1}$ or $x_k \geqslant x_{k+1}$ for all $k \in \mathbb{N}$. Then, $\{x_k\}_{k\in\mathbb{N}}$ converges.*

2.8 Compact Sets

A set C is said to be *compact* if every collection of open subsets of C whose union contains C can be reduced to a finite subcollection whose union also contains C.

Unlike in some metric spaces, the following characterization of compact sets holds in $\mathbb{E}^n$.

Theorem 2.8.1. *A set in $\mathbb{E}^n$ is compact if and only if it is closed and bounded.*

A proof of the last theorem can be found on page 40 of [60]. A subset of a compact set may or may not be compact. The next result describes a condition which guarantees that the property of compactness is passed down from a compact set to its subset; a proof of this result can be found on pages 37 and 38 of [50].

Theorem 2.8.2. *Closed subsets of compact sets are compact.*

A proof of the claim below can be found on page 1 of [53].

Proposition 2.8.3. *The finite union of compact sets is compact.*

The following theorem combines the concepts of closed and compact sets to provide a property for Minkowski sums; see page 43 of [60] for its proof.

Theorem 2.8.4. *Let $A \subseteq \mathbb{E}^n$ be compact and $B \subseteq \mathbb{E}^n$ be closed. Then, $A + B$ is closed.*

Let A be a non-empty subset of $\mathbb{E}^n$. For each $\mathbf{x} \in \mathbb{E}^n$, the *distance* between A and $\mathbf{x}$ is defined and denoted by

$$\mathrm{d}(A, \mathbf{x}) = \inf \{\|\mathbf{a} - \mathbf{x}\| \mid \mathbf{a} \in A\}.$$

In general, the infimum in the definition above cannot be replaced with minimum. However, the theorem below provides the conditions necessary for this replacement to occur; its proof can be found on pages 46 and 47 of [60].

Theorem 2.8.5. *If $A, B \in \mathbb{E}^n$ are non-empty sets where A is closed and B is compact, then there exist $\mathbf{a_0} \in A$ and $\mathbf{b_0} \in B$ such that*

$$\|\mathbf{a_0} - \mathbf{b_0}\| = \inf_{\mathbf{a} \in A, \mathbf{b} \in B} \|\mathbf{a} - \mathbf{b}\|.$$

Note that the points $\mathbf{a}_0$ and $\mathbf{b}_0$ are called the *nearest* points of A and B; they are not necessarily unique.

2.9 Continuous functions

Let $f : X \to Y$ be a function. If $S \subseteq Y$, then the set of all elements of X whose images under f lie in S is called the *pre-image* of S under f and is denoted by $f^{-1}(S)$. Note that when f is a bijection, the pre-image f^{-1} coincides with the inverse of f. A function $f : X \to Y$ is said to be *continuous* if for each open subset U of Y, the pre-image $f^{-1}(U)$ is an open subset of X.

A generalized version of the Extreme Value Theorem from Calculus is presented below; its proof can be found on page 174 of [42].

Extreme Value Theorem. *Let $f : X \to Y$ be a continuous function where Y is an ordered set in the order topology. If X is compact, then there exists elements $c, d \in X$ such that $f(c) \leqslant f(x) \leqslant f(d)$ for every $x \in X$.*

Another useful fact about affine transformations is stated in the following lemma, which is proved on pages 44 and 45 of [60].

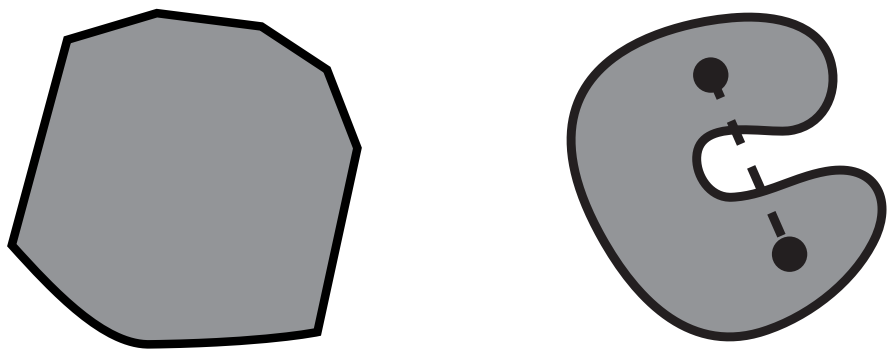
Lemma 2.9.1. *Affine transformations are continuous.*

The next theorem describes the action of continuous functions on compact sets; see page 30 of [37] for its proof.

Theorem 2.9.2. *Let $f : X \to Y$ be a continuous function and let $A \subseteq X$ be compact. Then, the image of A under f, $f(A)$, is compact.*

2.10 Convex Sets

A set is called *convex* if for any two vectors in the set, the line segment joining the two vectors is also contained by the set. Explicitly, a set C in $\mathbb{E}^n$ is convex if $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in C$ for any vectors $\mathbf{x}, \mathbf{y} \in C$ and any scalar $\lambda \in \mathbb{R}$ where $1 \geqslant \lambda \geqslant 0$.



Convex set in $\mathbb{E}^2$ Non-convex set in $\mathbb{E}^2$

Figure 2.3

Below is a collection of important convex sets. Proofs of the last two facts can be found on page 50 of [60].

Properties 2.10.1.

- ***(i)*** *Affine sets are convex.*
- ***(ii)*** *Line segments are convex.*
- ***(iii)*** *Halfspaces are convex.*
- ***(iv)*** *Balls are convex.*

It follows from (i) of Properties 2.10.1 that singleton sets and the empty set are convex. Note that, like affine sets, convex sets have the following property; a proof of this property can be found on page 50 of [60].

Theorem 2.10.2. *The intersection of an arbitrary family of convex sets in* $\mathbb{E}^n$ *is convex.*

The next property describes the interaction between convexity, Minkowski addition and scalar multiplication; its proof can be found on page 51 of [60].

Theorem 2.10.3. *If* $A \subseteq \mathbb{E}^n$ *is convex and* $\lambda_1, \lambda_1, \ldots, \lambda_m \geqslant 0$*, then*

$$(\lambda_1 + \lambda_1 + \ldots + \lambda_m)A = \lambda_1 A + \lambda_2 A + \ldots + \lambda_m A.$$

A vector is said to be a *convex combination* of the vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}$ in $\mathbb{E}^n$ if it can be written as $\lambda_1\mathbf{x_1} + \lambda_2\mathbf{x_2} + \ldots + \lambda_m\mathbf{x_m}$. The following theorem about convex combinations is proved on page 50 of [60].

Theorem 2.10.4. *Let $\mathbf{c}_1, \ldots, \mathbf{c}_m$ be elements of a convex set C in $\mathbb{E}^n$. Then the convex combination $\lambda_1\mathbf{c}_1 + \ldots + \lambda_m\mathbf{c}_m$ belongs to C for $\lambda_1, \ldots, \lambda_m \geqslant 0$ with $\lambda_1 + \ldots + \lambda_m = 1$.*

The *convex hull* of any set A in $\mathbb{E}^n$, denoted by $\text{conv}(A)$ is the intersection of all convex sets in $\mathbb{E}^n$ containing A. Let $\lambda_1, \lambda_1, \ldots, \lambda_m \geqslant 0$ be scalars with the property that $\lambda_1 + \lambda_2 + \ldots + \lambda_m = 1$. Equivalently, the convex hull of any set A in $\mathbb{E}^n$ is the set of all convex combinations of the vectors in A (see page 55 of [60]).

Two useful facts about the convex hull are stated below; sketches of their proofs are outlined on page 54 of [60].

Proposition 2.10.5. *The convex hull of any set $S \subseteq \mathbb{E}^n$, $\text{conv}(S)$, is the smallest convex set containing S. Furthermore, if $S \subseteq \mathbb{E}^n$ is convex, then $\text{conv}(S) = S$.*

Proposition 2.10.6. *If $A \subseteq B$, then $\text{conv}(A) \subseteq \text{conv}(B)$.*

The theorem and corollary below describe the action of the convex hull on open and compact sets, respectively; the theorem is proved on pages 57 and 58 of [60].

Theorem 2.10.7. *The convex hull of an open set in $\mathbb{E}^n$ is open and the convex hull of a compact set in $\mathbb{E}^n$ is compact.*

Corollary 2.10.8. *The convex hull of a finite set in $\mathbb{E}^n$ is compact.*

A proof of the next statement can be found on page 61 of [60].

Theorem 2.10.9. *The relative interior of a non-empty convex set in $\mathbb{E}^n$ is non-empty.*

The following theorem, although elementary, is extremely useful; it is proved on page 5 of [51] and page 62 of [60].

Theorem 2.10.10. *Let $A \subseteq \mathbb{E}^n$ be convex. If $\mathbf{x} \in \text{int}(A)$ and $\mathbf{y} \in \text{cl}(A)$, then $[\mathbf{x}, \mathbf{y}) \subseteq \text{int}(A)$. Likewise, if $\mathbf{x} \in \text{relint}(A)$ and $\mathbf{y} \in \text{cl}(A)$, then $[\mathbf{x}, \mathbf{y}) \subseteq \text{relint}(A)$.*

See page 61 of [60] for a proof of the corollary.

Corollary 2.10.11. *Let $A \subseteq \mathbb{E}^n$ be convex. If $\mathbf{x} \in \text{int}(A)$ and $\mathbf{y} \in A$, then $[\mathbf{x}, \mathbf{y}) \subseteq \text{int}(A)$.*

See page 210 of [3] for an explanation of the corollary below.

Corollary 2.10.12. *Every ray (half line) emanating from an interior point of a convex body intersects the boundary of the convex body at exactly one point.*

It immediately follows from Corollary 2.10.12 that any line passing through an interior point of some convex body will intersect the boundary of the convex body at exactly two points.

Lemma 2.10.13. *Let A be a closed convex set. For any arbitrarily chosen $\mathbf{x}, \mathbf{y} \in \text{bd}(A)$, $[\mathbf{x}, \mathbf{y}] \subseteq \text{bd}(S)$ or $(\mathbf{x}, \mathbf{y}) \subseteq \text{int}(S)$.*

Corollary 2.10.14. *Let $A \subseteq \mathbb{E}^n$ be a convex body. Then, $\text{bd}(A)$ is not convex.*

The following statement is proved on page 62 of [60].

Theorem 2.10.15. *Let $S \subseteq \mathbb{E}^n$ be a convex set. Then $\text{int}(S)$, $\text{relint}(S)$ and $\text{cl}(S)$ are convex sets.*

See page 73 of [8] for a proof of the next theorem.

Theorem 2.10.16. *Let $K \subseteq \mathbb{E}^2$ be a convex body. Then, $\text{bd}(K)$ is a simple closed curve.*

2.10.1 Support Hyperplanes and Separating Hyperplanes

Let K be a closed bounded convex set in $\mathbb{E}^n$ and let $\mathbf{k} \in K$ be arbitrarily chosen. A hyperplane H *supports* K if $H \cap K \neq \varnothing$ and either $K \subseteq \overline{H}^+$ or $K \subseteq \overline{H}^-$. In addition, if $\mathbf{k} \in H \cap K$, then the hyperplane H is said to *support K at* $\mathbf{k}$. Any hyperplane that supports

K is called a *supporting hyperplane* of K. Let $H = \{\mathbf{x} \in \mathbb{E}^3 \mid \langle \mathbf{x}, \mathbf{u} \rangle = \lambda,\ \mathbf{u} \in \mathcal{S}^{n-1},\ \lambda \in \mathbb{R}\}$ be a supporting hyperplane of K where $K \subseteq \overline{H}^+$. Then, $-\mathbf{u}$ is an *outward normal vector* of H and $\overline{H}^+$ is called the *supporting halfspace* of K. Likewise, $\mathbf{u}$ is an outward normal vector of H if $H \cap K \neq \varnothing$ and $\overline{H}^-$ is the supporting halfspace of K.

The following theorem is fundamental to the study of convex, discrete geometry; its proof can be found on pages 31 to 38 of [39].

Theorem 2.10.1.1. *Through each boundary point,* $\mathbf{x}$*, of a closed, convex set* C *in* $\mathbb{E}^n$ *there passes at least one hyperplane supporting* C *at* $\mathbf{x}$*.*

The question of how many parallel hyperplanes that support a convex body is answered in the next theorem; a proof of the theorem can be found on page 8 of [19].

Theorem 2.10.1.2. *Let* $C \subseteq \mathbb{E}^n$ *be some convex body. Then for each (closed affine) hyperplane* H *in* $\mathbb{E}^n$ *there exist exactly two supporting hyperplanes of* C*, which are parallel to* H *in* $\mathbb{E}^n$*.*

Theorem 2.10.1.3. *Every closed bounded convex set* K *in* $\mathbb{E}^n$ *is the intersection of all its supporting half-spaces.*

Proposition 2.10.1.4. *Let* S *be a closed, convex set of* $\mathbb{E}^n$ *with non-empty interior. Suppose that the hyperplane* $H = \{\mathbf{x} \in \mathbb{E}^3 \mid \langle \mathbf{x}, \mathbf{u} \rangle = \lambda,\ \mathbf{u} \in \mathcal{S}^{n-1},\ \lambda \in \mathbb{R}\}$ *meets* S *but does not support* S*. Then,* $H \cap \operatorname{int}(S) \neq \varnothing$*.*

Proof. Suppose H does not support S but $S \cap H \neq \varnothing$. It follows that $S \nsubseteq \overline{H}^-$ and $S \nsubseteq \overline{H}^+$. This means that there exists elements $\mathbf{s}_1, \mathbf{s}_2 \in S$ such that $\mathbf{s}_1 \in H^- = \{\mathbf{x} \in \mathbb{E}^3 \mid \langle \mathbf{x}, \mathbf{u} \rangle < \lambda,\ \mathbf{u} \in \mathcal{S}^{n-1},\ \lambda \in \mathbb{R}\}$ and $\mathbf{s}_2 \in H^+ = \{\mathbf{x} \in \mathbb{E}^3 \mid \langle \mathbf{x}, \mathbf{u} \rangle > \lambda,\ \mathbf{u} \in \mathcal{S}^{n-1},\ \lambda \in \mathbb{R}\}$. By convexity, $[\mathbf{s}_1, \mathbf{s}_2] \subseteq S$. Let

$$\mu = \frac{\lambda - \langle \mathbf{s}_1, \mathbf{u} \rangle}{\langle \mathbf{s}_2, \mathbf{u} \rangle - \langle \mathbf{s}_2, \mathbf{u} \rangle}.$$

Then,

$$\langle \mathbf{s}_1 + \mu\left(\mathbf{s}_2 - \mathbf{s}_1\right), \mathbf{u} \rangle = \langle \mathbf{s}_1, \mathbf{u} \rangle + \frac{\lambda - \langle \mathbf{s}_1, \mathbf{u} \rangle}{\langle \mathbf{s}_2, \mathbf{u} \rangle - \langle \mathbf{s}_1, \mathbf{u} \rangle}\Big(\langle \mathbf{s}_2, \mathbf{u} \rangle - \langle \mathbf{s}_1, \mathbf{u} \rangle\Big)$$

$$= \lambda.$$

Notice that

$$\langle \mathbf{s}_2, \mathbf{u} \rangle - \langle \mathbf{s}_1, \mathbf{u} \rangle > \lambda - \langle \mathbf{s}_1, \mathbf{u} \rangle > 0. \tag{$\star$}$$

It follows that, $\left(\langle \mathbf{s}_2, \mathbf{u} \rangle - \langle \mathbf{s}_1, \mathbf{u} \rangle \right)^2 > 0$. Moreover,

$$\frac{1}{\langle \mathbf{s}_2, \mathbf{u} \rangle - \langle \mathbf{s}_1, \mathbf{u} \rangle} \Big(\langle \mathbf{s}_2, \mathbf{u} \rangle - \langle \mathbf{s}_1, \mathbf{u} \rangle \Big)^2 = \langle \mathbf{s}_2, \mathbf{u} \rangle - \langle \mathbf{s}_1, \mathbf{u} \rangle > 0.$$

This means

$$\frac{1}{\langle \mathbf{s}_2, \mathbf{u} \rangle - \langle \mathbf{s}_1, \mathbf{u} \rangle} > 0.$$

Combine this inequality with ($\star$) to get that $0 < \mu < 1$.

Therefore, $\mathbf{s}_1 + \mu (\mathbf{s}_2 - \mathbf{s}_1) \in (\mathbf{s}_1, \mathbf{s}_2) \cap H \subseteq [\mathbf{s}_1, \mathbf{s}_2] \cap H \subseteq S \cap H$.

Recall that since S is closed, it can be written as the union of the disjoint sets $\text{int}(S)$ and $\text{bd}(S)$. It follows that either $\mathbf{s}_1 + \mu (\mathbf{s}_2 - \mathbf{s}_1) \in \text{int}(S)$ or $\mathbf{s}_1 + \mu (\mathbf{s}_2 - \mathbf{s}_1) \in \text{bd}(S)$.

Suppose $\mathbf{s}_1 + \mu (\mathbf{s}_2 - \mathbf{s}_1) \in \text{int}(S)$. Then, there is nothing more to show.

Now, suppose that $\mathbf{s}_1 + \mu (\mathbf{s}_2 - \mathbf{s}_1) \in \text{bd}(S)$. Then, $[\mathbf{s}_1, \mathbf{s}_2], [\mathbf{s}_1, \mathbf{s}_2), (\mathbf{s}_1, \mathbf{s}_2] \nsubseteq \text{int}(S)$. This together with Theorem 2.10.10 and the convexity of $\text{int}(S)$ imply that neither $\mathbf{s}_1 \in \text{int}(S)$ nor $\mathbf{s}_2 \in \text{int}(S)$. Therefore, $[\mathbf{s}_1, \mathbf{s}_2] \subseteq \text{bd}(S)$. Since $\text{int}(S) \neq \varnothing$, there exists $\mathbf{s}' \in \text{int}(S)$.

If $\mathbf{s}' \in H$, then there is nothing more to show.

If $\mathbf{s}' \in H^-$, then let $\mu' = \dfrac{\lambda - \langle \mathbf{s}', \mathbf{u} \rangle}{\langle \mathbf{s}_2, \mathbf{u} \rangle - \langle \mathbf{s}', \mathbf{u} \rangle}$ and observe that, just like above,

$$\begin{aligned} \langle \mathbf{s}' + \mu' (\mathbf{s}_2 - \mathbf{s}'), \mathbf{u} \rangle &= \langle \mathbf{s}', \mathbf{u} \rangle + \frac{\lambda - \langle \mathbf{s}', \mathbf{u} \rangle}{\langle \mathbf{s}_2, \mathbf{u} \rangle - \langle \mathbf{s}', \mathbf{u} \rangle} \Big(\langle \mathbf{s}_2, \mathbf{u} \rangle - \langle \mathbf{s}', \mathbf{u} \rangle \Big) \\ &= \lambda. \end{aligned}$$

Moreover, a nearly identical proof to the one above used for μ will show that $0 < \mu' < 1$. This means that $\mathbf{s}' + \mu' (\mathbf{s}_2 - \mathbf{s}') \in (\mathbf{s}', \mathbf{s}_2) \cap H$. By Theorem 2.10.10, $[\mathbf{s}', \mathbf{s}_2) \subseteq \text{int}(S)$. Therefore, $\mathbf{s}' + \mu' (\mathbf{s}_2 - \mathbf{s}') \in \text{int}(S) \cap H$.

If $\mathbf{s}' \in H^+$, then let $\mu'' = \dfrac{\lambda - \langle \mathbf{s}_1, \mathbf{u} \rangle}{\langle \mathbf{s}', \mathbf{u} \rangle - \langle \mathbf{s}_1, \mathbf{u} \rangle}$ and notice that

$$\langle \mathbf{s}_1 + \mu'' (\mathbf{s}' - \mathbf{s}_1), \mathbf{u} \rangle = \langle \mathbf{s}_1, \mathbf{u} \rangle + \frac{\lambda - \langle \mathbf{s}_1, \mathbf{u} \rangle}{\langle \mathbf{s}', \mathbf{u} \rangle - \langle \mathbf{s}_1, \mathbf{u} \rangle} \Big(\langle \mathbf{s}', \mathbf{u} \rangle - \langle \mathbf{s}_1, \mathbf{u} \rangle \Big)$$

$$= \lambda.$$

A very similar proof to the one used above for μ will show that $0 < \mu'' < 1$. This means that $\mathbf{s}_1 + \mu''(\mathbf{s}' - \mathbf{s}_1) \in (\mathbf{s}_1, \mathbf{s}') \cap H$. Again, by Theorem 2.10.10, $(\mathbf{s}_1, \mathbf{s}'] \subseteq \text{int}(S)$. Thus, $\mathbf{s}_1 + \mu''(\mathbf{s}' - \mathbf{s}_1) \in \text{int}(S) \cap H$. ■

Theorem 2.10.1.5. *Let A and B in $\mathbb{E}^n$ be disjoint, non-empty convex sets. Then, there exists a hyperplane H in $\mathbb{E}^n$ that properly separates A and B.*

2.11 Cones

A non-empty set $S \subseteq \mathbb{E}^n$ is called a *cone* if for every $\lambda \geqslant 0$ and every $\mathbf{s} \in S$, $\lambda\mathbf{s} \in S$. All cones contain the origin and all cones are unbounded sets, except for the *trivial cone*: $\{\mathbf{o}\}$.

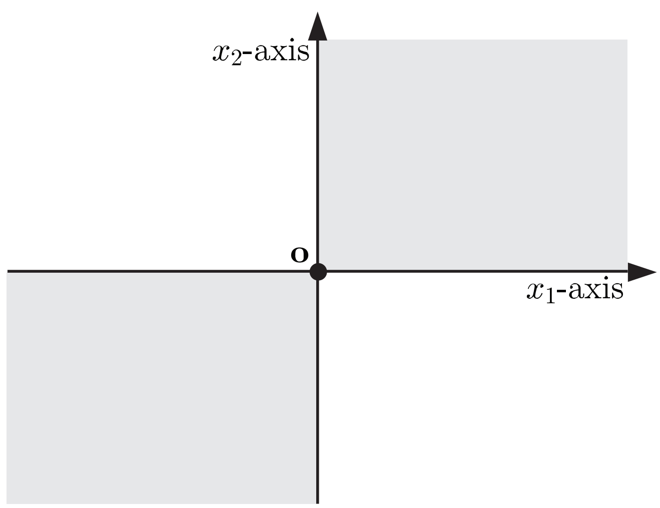


Figure 2.4: Not all cones are convex. For example, the set $\{(x_1, x_2) \mid x_1 x_2 \geqslant 0\}$ in $\mathbb{E}^2$ is a cone which is not convex.

The following theorem provides a condition for determining when a cone is convex. Its proof can be found on page 76 of [60].

Theorem 2.11.1. *Let S be a non-empty set in $\mathbb{E}^n$. Then, S is a convex cone if and only if $\lambda_1\mathbf{s}_1 + \lambda_2\mathbf{s}_2 \in S$ for all $\mathbf{s}_1, \mathbf{s}_2 \in S$ and $\lambda_1, \lambda_2 \geqslant 0$.*

The set $\{\mathbf{s}_0 + \lambda\mathbf{s} \mid \mathbf{s}_0 \in \mathbb{E}^n,\ \mathbf{s} \neq \mathbf{o},\ \lambda \geqslant 0\}$ is referred to as the *ray emanating from* $\mathbf{s}_0$ *with direction* $\mathbf{s}$. The ray together with the zero vector is a 1-dimensional convex cone. A cone can be expressed as a union rays; the *apex* of a cone is the point from whence the rays emanate. For example, the apex of the cone

$$\bigcup_{\mathbf{s} \in S} \{\lambda\mathbf{s} \mid \lambda \geqslant 0\}$$

is the origin. Let S be a cone whose apex is the origin and let $\mathbf{s}_0 \neq \mathbf{o}$. The translate of S by $\mathbf{s}_0$ together with the zero vector, $\{\mathbf{s}_0 + \lambda\mathbf{s} \mid \mathbf{s} \in S,\ \lambda \geqslant 0\} \cup \{\mathbf{o}\}$, is a cone with apex $\mathbf{s}_0$.

Chapter 3

The Blaschke Selection Theorem, Mazur's Finite Dimensional Density Theorem and the John - Löwner Theorem

3.1 The Blaschke Selection Theorem

Let $\mathcal{K}^n$ denote the set containing all non-empty, compact, convex sets in $\mathbb{E}^n$. The distance between any two elements K_1 and K_2 of $\mathcal{K}^n$ is denoted and defined by

$$\delta\left(K_1, K_2\right) = \max\left\{\max_{\mathbf{k}_1 \in K_1} \min_{\mathbf{k}_2 \in K_2} \|\mathbf{k}_1 - \mathbf{k}_2\| \ , \ \max_{\mathbf{k}_2 \in K_2} \min_{\mathbf{k}_1 \in K_1} \|\mathbf{k}_1 - \mathbf{k}_2\|\right\}.$$

The function $\delta : \mathcal{K}^n \times \mathcal{K}^n \rightarrow \mathcal{K}^n$ is called the *Hausdorff distance*. The Hausdorff distance between any two non-empty, compact convex sets in $\mathbb{E}^n$ has an equivalent formulation:

Proposition 3.1.1. *Let $K_1, K_2 \in \mathcal{K}^n$ be arbitrarily chosen. Then,*

$$\delta\left(K_1, K_2\right) = \min\left\{\lambda \geqslant 0 \mid K_1 \subseteq K_2 + \lambda \mathrm{B}\left(\mathbf{o}, 1\right), \ K_2 \subseteq K_1 + \lambda \mathrm{B}\left(\mathbf{o}, 1\right)\right\}.$$

A detailed proof of Proposition 3.1.2 can be found on p. 12 of [6].

Proposition 3.1.2. *The set $\mathcal{K}^n$ together with δ is a metric space.*

A sequence $\{X_i\}$ is called a *Cauchy sequence* in the metric space $(\mathcal{X}, \hat{\mathrm{d}})$ if for all $\varepsilon > 0$, there exists an integer N such that

$$\hat{\mathrm{d}}\left(X_i, X_j\right) < \varepsilon,$$

whenever $i, j \geqslant N$. A metric space $(\mathcal{X}, \hat{\mathrm{d}})$ is said to be *complete* if every Cauchy sequence in $\mathcal{X}$ converges. A proof for the following property of Cauchy sequences can be found on page 20 of [37].

Proposition 3.1.3. *Every Cauchy sequence is bounded.*

The following well-known theorem is needed in the proof of Lemma 3.1.4. The overall structure and the second case of its proof is due to [4]; the first case is due to [30].

Cantor Intersection Theorem. *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of non-empty, compact sets from $\mathbb{E}^n$ such that $C_{i+1} \subseteq C_i$ for all $i \in \mathbb{N}$. Then, the set*

$$\bigcap_{i=1}^{\infty} C_i$$

is a non-empty, compact set of $\mathbb{E}^n$.

Proof. First, it will be shown that $\bigcap_{i=1}^{\infty} C_i$ is compact.
Let $\mathbf{x}_1, \mathbf{x}_2 \in \bigcap_{i=1}^{\infty} C_i$ be arbitrarily chosen. Notice that $\bigcap_{i=1}^{\infty} C_i \subseteq C_i$, for each $i \in \mathbb{N}$. Each C_i is compact in $\mathbb{E}^n$ and therefore, each C_i bounded in $\mathbb{E}^n$. Since $\mathbf{x}_1$ and $\mathbf{x}_2$ also belong to C_i, it follows that there exists a real number M such that $\|\mathbf{x}_1 - \mathbf{x}_2\| \leqslant M$. Thus, $\bigcap_{i=1}^{\infty} C_i$ is bounded.
Each C_i is compact in $\mathbb{E}^n$ and therefore, each C_i is closed in $\mathbb{E}^n$. It follows from Theorem 2.5.1 that $\bigcap_{i=1}^{\infty} C_i$ is closed in $\mathbb{E}^n$. Hence, $\bigcap_{i=1}^{\infty} C_i$ is compact in $\mathbb{E}^n$.

Finally, it will be shown that $\bigcap_{i=1}^{\infty} C_i \neq \varnothing$ in the following two cases.
Case 1: Suppose that not every C_i contains infinitely many points.
This means that there exists $C_{i_0} \in \{C_i\}_{i\in\mathbb{N}}$ such that $0 < |C_{i_0}| < \infty$. Since C_i is non-empty and $C_{i+1} \subseteq C_i$ for all $i \in \mathbb{N}$, it follows that

$$\infty > |C_{i_0}| \geqslant |C_{i_0+1}| \geqslant \ldots > 0.$$

The sequence $\{|C_i|\}_{i\geqslant i_0}$ is a bounded monotone decreasing sequence of integers. By Theorem 2.7.3, the sequence $\{|C_i|\}_{i\geqslant i_0}$ converges; denote the number to which it converges by L. Suppose for a contradiction that $L \in \mathbb{R}\backslash\mathbb{Z}$. Let $\varepsilon = \min\left\{\lceil L\rceil - L, L - \lfloor L\rfloor\right\}$. Recall that $\lceil L\rceil = \min\{n \in \mathbb{Z} \mid n \geqslant L\}$ and $\lfloor L\rfloor = \max\{m \in \mathbb{Z} \mid m \leqslant L\}$. It follows that

$$0 < \lceil L\rceil - L < 1 \quad \text{and} \quad 0 < L - \lfloor L\rfloor < 1.$$

Since $|C_i| \in \mathbb{Z}$ for all $i \geqslant i_0$, $|C_i|$ cannot be any closer to L than the integers $\lceil L \rceil$ or $\lfloor L \rfloor$. Therefore,

$$\big| |C_i| - L \big| \geqslant \lceil L \rceil - L \quad \text{and} \quad \big| |C_i| - L \big| \geqslant L - \lfloor L \rfloor,$$

for all $i \geqslant i_0$. Thus, $\big| |C_i| - L \big| \geqslant \varepsilon$, for all $i \geqslant i_0$. This is a contradiction. Hence, $L \in \mathbb{Z}$.

Like above, let $\varepsilon = \min \big\{ \lceil L \rceil - L, L - \lfloor L \rfloor \big\}$. Since the sequence $\{|C_i|\}_{i \geqslant i_0}$ converges to $L \in \mathbb{Z}$, there exists $N \in \mathbb{N}$ such that

$$\big| |C_i| - L \big| < \varepsilon,$$

for all $i > N$. Recall from above that $0 < \varepsilon < 1$. The minimum distance between two distinct integers is 1. Therefore, $\big| |C_i| - L \big| = 0$, for all $i > N$. This means that $|C_i| = L$, for all $i > N$. This implies two things: $L \neq 0$, since each $C_i \neq \varnothing$ and $C_i = C_{N+1}$ for all $i > N$, since $C_{i+1} \subseteq C_i$ for all $i \in \mathbb{N}$. Hence,

$$\bigcap_{i=1}^{\infty} C_i = C_{N+1} \neq \varnothing.$$

Case 2: Suppose that each C_i has infinitely many points.

Let $A = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ where $\mathbf{x}_i \in C_i$. Since $C_{i+1} \subseteq C_i$ for all $i \in \mathbb{N}$, it follows that $A \subseteq C_1$. Recall that C_1 is bounded. This together with the Bolzano Weierstrass Theorem implies that A contains a convergent subsequence; denote the point to which the subsequence converges by $\mathbf{x}$.

By definition, this means that every open neighbourhood of $\mathbf{x}$ intersects A at some point other than $\mathbf{x}$. In fact, every open neighbourhood of $\mathbf{x}$ contains infinitely many points of A. To see this, suppose for a contradiction that it is not so. This means that every open neighbourhood of $\mathbf{x}$ contains only a finite number of points from A which are distinct from $\mathbf{x}$; denote these points by $\mathbf{a}_1, \mathbf{a}_1, \ldots, \mathbf{a}_m$. Let $r = \min\{\|\mathbf{a}_1 - \mathbf{x}\|, \|\mathbf{a}_2 - \mathbf{x}\|, \ldots, \|\mathbf{a}_m - \mathbf{x}\|\}$. Notice that $r > 0$. Then, the open neighbourhood $\mathrm{B}\left(\mathbf{x}, \dfrac{r}{2}\right)$ of $\mathbf{x}$ has empty intersection with A. This is a contradiction.

Certainly, $\{\mathbf{x}_i, \mathbf{x}_{i+1}, \ldots\} \subseteq A \cap C_i$ for each $i \in \mathbb{N}$. To see that any open neighbourhood of $\mathbf{x}$ has non-empty intersection with C_i for all $i \in \mathbb{N}$, suppose for a contradiction that there exists an open neighbourhood U of $\mathbf{x}$ such that $U \cap C_i = \varnothing$ for some $i \in \mathbb{N}$. Then, $\{\mathbf{x}_i, \mathbf{x}_{i+1}, \ldots\} \cap U = \varnothing$. Notice that $\{\mathbf{x}_i, \mathbf{x}_{i+1}, \ldots\} = A \backslash \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{i-1}\}$ and recall that $A \cap U \neq \varnothing$. It follows that $A \cap U \subseteq \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{i-1}\}$, which means that $|A \cap U| \leqslant i - 1$ for some $i \in \mathbb{N}$. This contradicts that $|A \cap U| \nless \infty$.

By Theorem 2.5.3, $\mathbf{x} \in \mathrm{cl}\,(C_i)$ for all $i \in \mathbb{N}$. Since each C_i is closed, $\mathbf{x} \in C_i$ for all $i \in \mathbb{N}$. Thus, $\mathbf{x} \in \bigcap_{i=1}^{\infty} C_i$. This means that $\bigcap_{i=1}^{\infty} C_i \neq \varnothing$.

■

The lemma below is required in the proof of Theorem 3.1.5. The proof of the lemma is due to [51].

Lemma 3.1.4. *Let $\{K_i\}_{i \in \mathbb{N}}$ be a sequence from $\mathcal{K}^n$ such that $K_{i+1} \subseteq K_i$ for all $i \in \mathbb{N}$. Then,*

$$\delta\left(K_i\,, \bigcap_{i=1}^{\infty} K_i\right) \to 0$$

as $i \to \infty$.

Proof. It follows from Cantor Intersection Theorem and Theorem 2.10.2 that $\bigcap\limits_{i=1}^{\infty} K_i \in \mathcal{K}^n$. For simplicity, denote $\bigcap_{i=1}^{\infty} K_i$ by K. To show that $\delta\,(K_i\,, K) \to 0$ as $i \to \infty$, suppose for a contradiction that $\delta\,(K_i\,, K) \nrightarrow 0$ as $i \to \infty$. This means that there exists $\varepsilon > 0$ such that $K_i \nsubseteq K + \varepsilon\mathrm{B}\,(\mathbf{o}, 1)$ for all $i \in \mathbb{N}$. Let $A_i = K_i \backslash \mathrm{int}\big(K + \varepsilon\mathrm{B}\,(\mathbf{o}, 1)\big)$. Since K_i is non-empty and $K_i \nsubseteq K + \varepsilon\mathrm{B}\,(\mathbf{o}, 1)$, there exists $\mathbf{x} \in K_i$ such that $\mathbf{x} \notin K + \varepsilon\mathrm{B}\,(\mathbf{o}, 1)$ for all $i \in \mathbb{N}$. This means $\mathbf{x} \notin \mathrm{int}\big(K + \varepsilon\mathrm{B}\,(\mathbf{o}, 1)\big)$, since $\mathrm{int}\big(K + \varepsilon\mathrm{B}\,(\mathbf{o}, 1)\big) \subseteq K + \varepsilon\mathrm{B}\,(\mathbf{o}, 1)$. Therefore, $\mathbf{x} \in A_i$ for all $i \in \mathbb{N}$. This means that A_i is non-empty for all $i \in N$. Notice that $A_i \subseteq K_i$ for all $i \in \mathbb{N}$. Therefore, A_i is bounded in $\mathbb{E}^n$ for all $i \in \mathbb{N}$. Also, notice that

$$\begin{aligned}
A_i &= K_i \backslash \mathrm{int}\big(K + \varepsilon\mathrm{B}\,(\mathbf{o}, 1)\big)) \\
&= K_i \cap \Big(\mathbb{E}^n \backslash \mathrm{int}\big(K + \varepsilon\mathrm{B}\,(\mathbf{o}, 1)\big)\Big) \\
&= K_i \cap \mathrm{cl}\Big(\mathbb{E}^n \backslash \big(K + \varepsilon\mathrm{B}\,(\mathbf{o}, 1)\big)\Big).
\end{aligned}$$

Each K_i is closed and by Theorem 2.5.1, $\mathrm{cl}\Big(\mathbb{E}^n \backslash \big(K + \varepsilon\mathrm{B}\,(\mathbf{o}, 1)\big)\Big)$ is closed in $\mathbb{E}^n$. Thus, each A_i is closed in $\mathbb{E}^n$. Hence, A_i is compact. Moreover, $A_{i+1} \subseteq A_i$ for all $i \in \mathbb{N}$. Therefore, it follows from Cantor Intersection Theorem that $\bigcap_{i=1}^{\infty} A_i$ is non-empty.

It will be helpful to notice that $K \subseteq \operatorname{int}\big(K + \varepsilon \mathrm{B}(\mathbf{o}, 1)\big)$. To see this, begin by arbitrarily selecting $\mathbf{x} \in K$. Then, $\mathrm{B}(\mathbf{x}, \varepsilon) = \mathbf{x} + \varepsilon \mathrm{B}(\mathbf{o}, 1) \subseteq \bigcup_{\mathbf{k} \in K} \mathbf{k} + \varepsilon \mathrm{B}(\mathbf{o}, 1) = K + \varepsilon \mathrm{B}(\mathbf{o}, 1)$. By definition, $\mathbf{x} \in \operatorname{int}\big(K + \varepsilon \mathrm{B}(\mathbf{o}, 1)\big)$. This, in particular, implies that $K \backslash \operatorname{int}\big(K + \varepsilon \mathrm{B}(\mathbf{o}, 1)\big) = \varnothing$. Now, observe that

$$
\begin{aligned}
K \cap \bigcap_{i=1}^{\infty} A_i &= K \cap \bigcap_{i=1}^{\infty} \Big(K_i \backslash \operatorname{int}\big(K + \varepsilon \mathrm{B}(\mathbf{o}, 1)\big)\Big) \\
&= K \cap \Big(K_1 \backslash \operatorname{int}\big(K + \varepsilon \mathrm{B}(\mathbf{o}, 1)\big) \cap \bigcap_{i=2}^{\infty} \Big(K_i \backslash \operatorname{int}\big(K + \varepsilon \mathrm{B}(\mathbf{o}, 1)\big)\Big)
\end{aligned}
$$

then, it follows from a basic set theory identity that

$$
\begin{aligned}
&= K_1 \cap \Big(K \backslash \operatorname{int}\big(K + \varepsilon \mathrm{B}(\mathbf{o}, 1)\big)\Big) \cap \bigcap_{i=2}^{\infty} \Big(K_i \backslash \operatorname{int}\big(K + \varepsilon \mathrm{B}(\mathbf{o}, 1)\big)\Big) \\
&= \varnothing \cap \bigcap_{i=2}^{\infty} \Big(K_i \backslash \operatorname{int}\big(K + \varepsilon \mathrm{B}(\mathbf{o}, 1)\big)\Big) = \varnothing.
\end{aligned}
$$

Since each $A_i \subseteq K_i$, it follows that $\bigcap_{i=1}^{\infty} A_i \subseteq K$. This is a contradiction. Hence,

$$
\delta\left(K_i \,, \bigcap_{i=1}^{\infty} K_i\right) \to 0,
$$

as $i \to \infty$. ∎

The following theorem plays an essential role in the proof of The Blaschke Selection Theorem. Its proof is also due to [51].

Theorem 3.1.5. *The metric space $(\mathcal{K}^n, \delta)$ is complete.*

Proof. Let $\{K_i\}_{i \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{K}^n$ and let

$$
A_m = \operatorname{cl}\left(\bigcup_{i=m}^{\infty} K_i\right) = \operatorname{cl}(K_m \cup K_{m+1} \cup \ldots).
$$

Since $K_{m+1} \cup K_{m+2} \cup \ldots \subseteq K_m \cup K_{m+1} \cup \ldots$, it follows from Corollary 2.5.6 that

$$
A_{m+1} = \operatorname{cl}(K_{m+1} \cup K_{m+2} \cup \ldots) \subseteq \operatorname{cl}(K_m \cup K_{m+1} \cup \ldots) = A_m.
$$

Notice that

$$K_i \subseteq K_m \cup K_{m+1} \cup \ldots \subseteq \mathrm{cl}\left(K_m \cup K_{m+1} \cup \ldots\right) = A_m, \tag{$\star$}$$

for any $i \geqslant m$ and since each $K_i \in \mathcal{K}^n$ is non-empty, it follows that A_m is non-empty. Also, each A_m is closed by Theorem 2.5.1.

Claim: $\bigcup\limits_{i=m}^{\infty} K_i$ is bounded for any $m \in \mathbb{N}$.

Let $\mathbf{k}', \mathbf{k}'' \in \bigcup_{i=m}^{\infty} K_i$ and $i^\star \in \{m, m+1, \ldots\}$ be arbitrarily chosen. Then, there exists K_i and K_j where $i, j \in \{m, m+1, \ldots\}$ such that $\mathbf{k}' \in K_i$ and $\mathbf{k}'' \in K_j$. By Proposition 3.1.3, there exists a real number $M \geqslant 0$ such that $\delta\left(K_p, K_q\right) \leqslant M$ for all $K_p, K_q \in \{K_i\}_{i \in \mathbb{N}}$. In particular, this means that $K_i \subseteq K_{i^\star} + M\mathrm{B}^n\left(\mathbf{o}, 1\right)$ and $K_j \subseteq K_{i^\star} + M\mathrm{B}^n\left(\mathbf{o}, 1\right)$. This implies that there exists $\mathbf{x}, \mathbf{y} \in K_{i^\star}$ and $\mathbf{b}_1, \mathbf{b}_2 \in \mathrm{B}^n\left(\mathbf{o}, 1\right)$ such that $\mathbf{k}' = \mathbf{x} + M\mathbf{b}_1$ and $\mathbf{k}'' = \mathbf{y} + M\mathbf{b}_2$. Observe that

$$\|\mathbf{k}' - \mathbf{k}''\| = \|\mathbf{x} + M\mathbf{b}_1 - (\mathbf{y} + M\mathbf{b}_2)\,\|$$

then, by the triangle inequality,

$$\leqslant \|\mathbf{x} - \mathbf{y}\| + M\|\mathbf{b}_1 - \mathbf{b}_2\|$$

then, using the triangle inequality again,

$$\begin{aligned} &\leqslant \|\mathbf{x} - \mathbf{y}\| + M\left(\|\mathbf{b}_1\| + |-1| \cdot \|\mathbf{b}_2\|\right) \\ &\leqslant \|\mathbf{x} - \mathbf{y}\| + 2M \end{aligned}$$

then, since each K_i is bounded there exists a real number $M_{i^\star} \geqslant 0$ such that

$$\leqslant M_{i^\star} + 2M.$$

To see that each A_m is bounded, let $\mathbf{a}, \mathbf{a}' \in A_m$ be arbitrarily chosen. It follows from Theorem 2.6.3 that there exists sequences $\{\mathbf{a}_i\}_{i \in \mathbb{N}}$ and $\{\mathbf{a}'_j\}_{j \in \mathbb{N}}$ whose elements belong to $\bigcup\limits_{i=m}^{\infty} K_i$ and which converge to $\mathbf{a}$ and $\mathbf{a}'$, respectively. Then,

$$\|\mathbf{a} - \mathbf{a}'\| = \left\|(\mathbf{a} - \mathbf{a}_i) + \left(\mathbf{a}'_j - \mathbf{a}'\right) + (\mathbf{a}_i - \mathbf{a}_j)\right\|$$

then, by the triangle inequality,

$$\leqslant |-1| \left\|\mathbf{a}_i - \mathbf{a}\right\| + \left\|\mathbf{a}'_j - \mathbf{a}'\right\| + \left\|\mathbf{a}_i - \mathbf{a}_j\right\|$$

then, since $\bigcup_{i=m}^{\infty} K_i$ is bounded,

$$\leqslant |-1| \left\|\mathbf{a}_i - \mathbf{a}\right\| + \left\|\mathbf{a}'_j - \mathbf{a}'\right\| + M_{i^\star} + 2M$$

then, since $\mathbf{a}_i \to \mathbf{a}$ and $\mathbf{a}'_j \to \mathbf{a}'$, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$< \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} + M = \varepsilon' + M_{i^\star} + 2M,$$

for all $i > N_1$ and $j > N_2$. Hence, each A_m is compact.

By Lemma 3.1.4, the sequence $\{A_m\}_{m\in\mathbb{N}}$ converges to $A = \bigcap_{i=1}^{\infty} A_i$ as $m \to \infty$. Then, for some $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $A_m \subseteq A + \varepsilon \mathrm{B}^n(\mathbf{o}, 1)$, for any $m > N$. It follows from $(\star)$ that $K_i \subseteq A_m \subseteq A + \varepsilon \mathrm{B}^n(\mathbf{o}, 1)$, for all $i \geqslant m > N$.

Since $\{K_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence, there exists $N' \in \mathbb{N}$ such that $K_j \subseteq K_i + \frac{\varepsilon}{3}\mathrm{B}^n(\mathbf{o}, 1)$ for all $i, j \geqslant N'$. Let $N^\star = \max\{N, N'\}$. It follows that for all $i, m \geqslant N^\star + 1$,

$$K_m \subseteq K_i + \frac{\varepsilon}{3}\mathrm{B}^n(\mathbf{o}, 1), \quad K_{m+1} \subseteq K_i + \frac{\varepsilon}{3}\mathrm{B}^n(\mathbf{o}, 1), \quad \ldots .$$

Therefore,

$$\bigcup_{j=m}^{\infty} K_j \subseteq K_i + \frac{\varepsilon}{3}\mathrm{B}^n(\mathbf{o}, 1).$$

It follows that

$$A_m = \mathrm{cl}\left(\bigcup_{j=m}^{\infty} K_j\right) \subseteq \mathrm{cl}\left(K_i + \frac{\varepsilon}{3}\mathrm{B}^n(\mathbf{o}, 1)\right) = K_i + \frac{\varepsilon}{3}\mathrm{B}^n[\mathbf{o}, 1] \subseteq K_i + \frac{\varepsilon}{2}\mathrm{B}^n(\mathbf{o}, 1), \qquad (\dagger)$$

for all $i, m \geqslant N^\star + 1$. As a consequence of Lemma 3.1.4, there exists $N'' \in \mathbb{N}$ such that

$$A \subseteq A_m + \frac{\varepsilon}{2}\mathrm{B}^n(\mathbf{o}, 1), \qquad (\ddagger)$$

for any $m > N''$. Let $\hat{N} = \max\{N^\star + 1, N''\}$. It follows from $(\ddagger)$, $(\dagger)$ and Proposition 2.1.1 that

$$A \subseteq A_m + \frac{\varepsilon}{2}\mathrm{B}^n(\mathbf{o}, 1) \subseteq K_i + \frac{\varepsilon}{2}\mathrm{B}^n(\mathbf{o}, 1) + \frac{\varepsilon}{2}\mathrm{B}^n(\mathbf{o}, 1)$$

then, by Properties 2.10.1 and Theorem 2.10.3,

$$= K_i + \varepsilon \mathrm{B}^n(\mathbf{o}, 1),$$

for all $i, m \geqslant \hat{N} + 1$. Hence, $\delta(K_i, A) < \varepsilon$ for any $i > \hat{N} + 1$.

■

A *closed n-cube* of $\mathbb{E}^n$ with centre $\mathbf{a}$ and side length $2R$ is defined by

$$\{\mathbf{z} \in \mathbb{E}^n \mid \max\{|z_1 - a_1|, |z_2 - a_2|, \ldots, |z_n - a_n|\} \leqslant R\}.$$

The proof of the well-known The Blaschke Selection Theorem is due to [35] and [51].

The Blaschke Selection Theorem. *From each bounded sequence of convex bodies one can select a subsequence converging to a convex body.*

Proof of Blaschke Selection Theorem.

Let $\{K_i\}_{i\in\mathbb{N}}$ be a bounded sequence whose elements belong to $\mathcal{K}^n$. This means that there exists a real number $M \geqslant 0$ such that $\delta(K, K^\star) \leqslant M$, for all $K, K^\star \in \{K_i\}_{i\in\mathbb{N}}$. Each K_i of the sequence $\{K_i\}_{i\in\mathbb{N}}$ is non-empty, since each $K_i \in \mathcal{K}^n$. Therefore, $\bigcup_{i=1}^{\infty} K_i \neq \varnothing$. Let $\mathbf{k} \in \bigcup_{i=1}^{\infty} K_i$ be arbitrarily chosen. Also, choose some $K_{i^\star} \in \{K_i\}_{i\in\mathbb{N}}$ arbitrarily. Since $K_{i^\star} \in \mathcal{K}^n$, it follows that $K_{i^\star}$ is compact and thus, bounded. So, there exists a real number $M_{i^\star} \geqslant 0$ such that $\|\mathbf{k}_1 - \mathbf{k}_2\| \leqslant M_{i^\star}$, for any $\mathbf{k}_1, \mathbf{k}_2 \in K_{i^\star}$. Let

$$C = \left\{\mathbf{z} \in \mathbb{E}^n \mid \max\left\{|z_1|, \ldots, |z_n|\right\} \leqslant 2M + M_{i^\star} + \|\mathbf{k}\|\right\}.$$

Claim: $\displaystyle\bigcup_{i=1}^{\infty} K_i \subseteq C$.

Let $\mathbf{x} \in \bigcup_{i=1}^{\infty} K_i$ be arbitrarily chosen. Then, there exists $i, j \in \mathbb{N}$ such that $\mathbf{x} \in K_i$ and $\mathbf{k} \in K_j$. It follows from the boundedness of $\{K_i\}_{i\in\mathbb{N}}$ that $K_i \subseteq K_{i^\star} + M\mathrm{B}^n(\mathbf{o}, 1)$ and $K_j \subseteq K_{i^\star} + M\mathrm{B}^n(\mathbf{o}, 1)$. Therefore, there exists $\mathbf{k}_1, \mathbf{k}_2 \in K_{i^\star}$ and $\mathbf{b}_1, \mathbf{b}_2 \in \mathrm{B}^n(\mathbf{o}, 1)$ such that $\mathbf{x} = \mathbf{k}_1 + M\mathbf{b}_1$ and $\mathbf{k} = \mathbf{k}_2 + M\mathbf{b}_2$. Observe that

$$\max\{\,|x_1|, \ldots, |x_n|\,\} \leqslant \sqrt{\sum_{i=1}^{n} |x_i|^2} = \|\mathbf{x}\|$$

then, by the triange inequality,

$$\leqslant \|\mathbf{x} - \mathbf{k}\| + \|\mathbf{k}\|$$
$$= \|\mathbf{k}_1 + M\mathbf{b}_1 - (\mathbf{k}_2 + M\mathbf{b}_2)\| + \|\mathbf{k}\|$$

then, by the triangle inequality,

$$\leqslant \|\mathbf{k}_1 - \mathbf{k}_2\| + M\left(\|\mathbf{b}_1\| + |-1| \cdot \|\mathbf{b}_2\|\right) + \|\mathbf{k}\|$$
$$\leqslant M_{i^\star} + 2M + \|\mathbf{k}\|.$$

This implies that $\mathbf{x} \in C$.

Note that C is a cube with edge length $\widehat{M} = 2 \cdot (2M + M_{i^\star} + \|\mathbf{k}\|)$. It follows from the claim that the sequence $\{K_i\}_{i\in\mathbb{N}}$ is contained in the cube C. Sub-divide each edge of C into 2^m equal parts to create 2^{mn} closed sub-cubes of C each with edge length $\dfrac{\widehat{M}}{2^m}$, whose union is equal to C; denote the collection of these sub-cubes by $C_m = \left\{C_{m_j}\right\}_{m_j \in I}$ where $I = \{1, \ldots, 2^{mn}\}$.

If $K \in \{K_i\}_{i\in\mathbb{N}}$ has non-empty intersection with a collection of sub-cubes $\left\{C_{m_j}\right\}_{m_j\in S}$ where $S \subseteq \{1, \ldots, 2^{mn}\}$ from C_m, then $\bigcup_{m_j\in S} C_{m_j}$ is said to be an $|S|$-*minimal covering* of K.

For $m = 1$, there are only finitely many possible minimal coverings for the elements of the sequence $\{K_i\}_{i\in\mathbb{N}}$ with the 2^n sub-cubes of C_1; namely, $2^{2^n} - 1$ possible minimal coverings. By the Infinite Pigeonhole Principle (see [55]), there exists infinitely many elements of the sequence $\{K_i\}_{i\in\mathbb{N}}$ with the same minimal covering. Denote this subsequence of $\{K_i\}_{i\in\mathbb{N}}$ whose elements have the same minimal covering, by $\{K_{1_i}\}_{i\in\mathbb{N}}$.

Similarly, the sequence $\{K_{1_i}\}_{i\in\mathbb{N}}$ contains an infinite subsequence $\{K_{2_i}\}_{i\in\mathbb{N}}$ whose elements have the same minimal covering by sub-cubes of C_2.

Continue in this way to obtain a sequence $\{K_{m_i}\}_{i\in\mathbb{N}}$ whose elements have the same minimal covering by sub-cubes of C_m and which is a subsequence of $\{K_{p_i}\}_{i\in\mathbb{N}}$, for any fixed $m \in \mathbb{N}$ and for any integer $p \leqslant m$.

Claim: For any $i, j \in \mathbb{N}$ and any fixed integers $p \leqslant m$, $\delta\left(K_{p_i}, K_{m_j}\right) \leqslant \dfrac{\widehat{M}\sqrt{n}}{2^m}$.

Denote the sub-cubes which minimally cover K_{p_i} by $\left\{C_{p_q}\right\}_{q\in S}$ where $S \subseteq \{1, \ldots, 2^{pn}\}$. Since $p \leqslant m$, the element K_{m_j} belongs to a subsequence of the sequence to which K_{p_i} belongs and therefore, is also minimally covered by the same sub-cubes of C_p which minimally cover K_{p_i}. It follows that

$$K_{m_j} \subseteq \bigcup_{q\in S} C_{p_q}. \tag{$\star$}$$

Let $\mathbf{x} \in \bigcup_{q\in S} C_{p_q}$ be arbitrarily chosen. There exists at least one sub-cube $C^\star \in \bigcup_{q\in S} C_{p_q}$ such that $\mathbf{x} \in C^\star$. Since $C^\star$ is part of the minimal covering of K_{p_i}, $K_{p_i} \cap C^\star \neq \varnothing$. Both K_{p_i} and $C^\star$ are closed, so $K_{p_i} \cap C^\star$ is closed by Theorem 2.5.1. The set $\{\mathbf{x}\}$ is closed and bounded and thus, compact in $\mathbb{E}^n$. It follows from Theorem 2.8.5 that there exists $\mathbf{z} \in K_{p_i} \cap C^\star$ so that $\mathrm{d}\left(\mathbf{x}, K_{p_i} \cap C^\star\right) = \|\mathbf{x} - \mathbf{z}\|$. Observe that

$$\begin{aligned}\|\mathbf{x} - \mathbf{z}\| = \mathrm{d}\left(\mathbf{x}, K_{p_i} \cap C^\star\right) &= \inf\left\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{y} \in K_{p_i}\right\} \\ &\leqslant \max\left\{\|\mathbf{c}_1 - \mathbf{c}_2\| \mid \mathbf{c}_1, \mathbf{c}_2 \in C^\star\right\} = \frac{\widehat{M}\sqrt{n}}{2^p}.\end{aligned}$$

Therefore,

$$\mathbf{x} \in \mathrm{B}^n\left(\mathbf{z}, \frac{\widehat{M}\sqrt{n}}{2^p}\right) = \mathbf{z} + \frac{\widehat{M}\sqrt{n}}{2^p} \cdot \mathrm{B}^n\left(\mathbf{o}, 1\right) \subseteq K_{p_i} + \frac{\widehat{M}\sqrt{n}}{2^p} \cdot \mathrm{B}^n\left(\mathbf{o}, 1\right).$$

It follows that

$$\bigcup_{q\in S} C_{p_q} \subseteq K_{p_i} + \frac{\widehat{M}\sqrt{n}}{2^p} \cdot \mathrm{B}^n\left(\mathbf{o}, 1\right).$$

This together with (($\star$)) implies that

$$K_{m_j} \subseteq K_{p_i} + \frac{\widehat{M}\sqrt{n}}{2^p} \cdot \mathrm{B}^n\left(\mathbf{o}, 1\right).$$

A similar argument can be used to show that $K_{p_i} \subseteq K_{m_j} + \frac{\widehat{M}\sqrt{n}}{2^p} \cdot \mathrm{B}^n\left(\mathbf{o}, 1\right)$. Hence, $\delta\left(K_{p_i}, K_{m_j}\right) \leqslant \frac{\widehat{M}\sqrt{n}}{2^p}$ for any $i, j \in \mathbb{N}$ and any fixed integers $p \leqslant m$.

It follows from the Claim that for any fixed integers $p \leqslant m$,

$$\delta\left(K_{p_p}, K_{m_m}\right) \leqslant \frac{\widehat{M}\sqrt{n}}{2^p}.$$

Thus, for any $\varepsilon > 0$, let $N = \dfrac{\widehat{M}\sqrt{n}}{\varepsilon}$. Recall that $\dfrac{1}{2^p} < \dfrac{1}{p}$ for all $p \in \mathbb{N}$. Then, whenever $m, p > N$,

$$\begin{aligned}\delta\left(K_{p_p}, K_{m_m}\right) &\leqslant \frac{\widehat{M}\sqrt{n}}{2^p} \\ &< \frac{\widehat{M}\sqrt{n}}{p} < \frac{\widehat{M}\sqrt{n}}{N} = \varepsilon.\end{aligned}$$

Thus, $\{K_{m_m}\}_{m\in\mathbb{N}}$ is a Cauchy sequence. By Theorem 3.1.5, $\{K_{m_m}\}_{m\in\mathbb{N}}$ converges to an element of $\mathcal{K}^n$. ■

The following Theorem describes how a convergent sequence of convex bodies can be expressed as a convergent sequence of points. Its proof can be found on page 63 of [51].

Theorem 3.1.6. *A sequence $\{K_i\}_{i\in\mathbb{N}}$ of convex bodies converges to K a convex body if and only if*

(i) Each element in K is the limit point of a sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \in K_i$.

(ii) The limit point of any convergent sequence $\{k_{i_j}\}_{j\in\mathbb{N}}$ with $k_{i_j} \in K_{i_j}$ belongs to K.

3.2 Mazur's Finite Dimensional Density Theorem

A set S is *dense* in X if $\mathrm{cl}(S) = X$. In contrast, a set is S called *nowhere dense* in X if $\mathrm{int}(\mathrm{cl}(S)) = \varnothing$. The countable union of nowhere dense sets is a *meagre* set.

The elements from the boundary of a convex body K can be classified into two disjoint set of points as follows. Let $\mathbf{k} \in \mathrm{bd}(K)$ be an arbitrarily chosen. It follows from Theorem 2.10.1.1 that there exists at least one hyperplane H through $\mathbf{k}$, which supports K. If H is unique, the boundary point $\mathbf{k}$ is said to be a *smooth point*. Otherwise, the boundary point $\mathbf{k}$ is called a *singular point*.

Let H be a supporting hyperplane of a convex body K at the point $\mathbf{k}' \in \text{bd}(K)$. Then, $H = \{\mathbf{x} \in \mathbb{E}^3 \mid \langle \mathbf{x}, \mathbf{u} \rangle = \langle \mathbf{k}', \mathbf{u} \rangle, \ \mathbf{u} \in \mathcal{S}^{n-1}\}$ and the set

$$\mathrm{N}_K(\mathbf{k}') = \left\{\lambda \mathbf{u} \mid \langle \mathbf{k}, \lambda \mathbf{u} \rangle \leqslant \langle \mathbf{k}', \lambda \mathbf{u} \rangle, \ \forall \ \mathbf{k} \in K, \ \lambda \in \mathbb{R}, \ \mathbf{u} \in S^{n-1}\right\}$$

contains all outward normal vectors of each supporting hyperplane at the point $\mathbf{k}' \in K$ together with the zero vector. If $\mathbf{k}'$ is a smooth boundary point of K, then $\mathrm{N}_K(\mathbf{k}')$ is a ray emanating from $\mathbf{o}$ and therefore, $\mathrm{N}_K(\mathbf{k}')$ is one-dimensional. If $\mathbf{k}'$ is a singular boundary point of K, then the dimension of $\mathrm{N}_K(\mathbf{k}')$ is at least two (see p. 70 of [26]). The set $\mathrm{N}_K(\mathbf{k}')$ is called the *normal cone of K at $\mathbf{k}'$*; the lemma below explains why the set is so named. The proof of the lemma is due to [26].

Lemma 3.2.1. *The set $\mathrm{N}_K(\mathbf{k}')$ is a closed, convex cone.*

Proof. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathrm{N}_K(\mathbf{k}')$ be arbitrarily chosen. Then,

$$\langle \mathbf{x}, \mathbf{v}_1 \rangle \leqslant \langle \mathbf{k}', \mathbf{v}_1 \rangle \quad \text{and} \quad \langle \mathbf{x}, \mathbf{v}_2 \rangle \leqslant \langle \mathbf{k}', \mathbf{v}_2 \rangle$$

for all $\mathbf{x} \in K$. For any arbitrarily chosen real numbers $\lambda_1, \lambda_2 \geqslant 0$,

$$\lambda_1 \langle \mathbf{x}, \mathbf{v}_1 \rangle \leqslant \lambda_1 \langle \mathbf{k}', \mathbf{v}_1 \rangle \quad \text{and} \quad \lambda_2 \langle \mathbf{x}, \mathbf{v}_2 \rangle \leqslant \lambda_2 \langle \mathbf{k}', \mathbf{v}_2 \rangle$$

and thus,

$$\lambda_1 \langle \mathbf{x}, \mathbf{v}_1 \rangle + \lambda_2 \langle \mathbf{x}, \mathbf{v}_2 \rangle \leqslant \lambda_1 \langle \mathbf{k}', \mathbf{v}_1 \rangle + \lambda_2 \langle \mathbf{k}', \mathbf{v}_2 \rangle$$

for all $\mathbf{x} \in K$. Use the properties of the inner product to get that

$$\langle \mathbf{x}, \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \rangle \leqslant \langle \mathbf{k}', \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \rangle$$

for all $\mathbf{x} \in K$. This means that $\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \in \mathrm{N}_K(\mathbf{k}')$. It follows from Theorem 2.11.1 that $\mathrm{N}_K(\mathbf{k}')$ is a convex cone.

Let $\{\mathbf{v}_i\}_{i \in \mathbb{N}}$ be a convergent sequence whose elements belong to $\mathrm{N}_K(\mathbf{k}')$; denote the point to which the sequence converges by $\mathbf{v} \in \mathbb{E}^n$. Let $\mathbf{x} \in K$ be arbitrarily chosen. It follows that

$$\langle \mathbf{x}, \mathbf{v}_i \rangle \leqslant \langle \mathbf{k}', \mathbf{v}_i \rangle$$

for any $i \in \mathbb{N}$. Notice that

$$|\langle \mathbf{x}, \mathbf{v}_i \rangle - \langle \mathbf{x}, \mathbf{v} \rangle| = |\langle \mathbf{x}, \mathbf{v}_i - \mathbf{v} \rangle|,$$

then, by the Cauchy Schwarz inequality,

$$\leqslant \|\mathbf{x}\| \|\mathbf{v}_i - \mathbf{v}\|$$

and if $\mathbf{x} \neq \mathbf{o}$, then

$$< \|\mathbf{x}\| \frac{\varepsilon}{\|\mathbf{x}\|} = \varepsilon$$

since $\mathbf{v}_i \to \mathbf{v}$ as $i \to \infty$. If $\mathbf{x} = \mathbf{o}$, then $\langle \mathbf{x}, \mathbf{v}_i \rangle = \langle \mathbf{x}, \mathbf{v} \rangle$ for all $i \in \mathbb{N}$; this would mean that the sequence of real numbers $\{\langle \mathbf{x}, \mathbf{v}_i \rangle\}_{i \in \mathbb{N}}$ is a constant sequence. In either case, the sequence of real numbers $\{\langle \mathbf{x}, \mathbf{v}_i \rangle\}_{i \in \mathbb{N}}$ converges to the real number $\langle \mathbf{x}, \mathbf{v} \rangle$. A nearly identical argument can be used to show that the sequence of real numbers $\{\langle \mathbf{k}', \mathbf{v}_i \rangle\}_{i \in \mathbb{N}}$ converges to the real number $\langle \mathbf{k}', \mathbf{v} \rangle$. By Theorem 2.6.4,

$$\langle \mathbf{x}, \mathbf{v} \rangle \leqslant \langle \mathbf{k}', \mathbf{v} \rangle$$

for all $\mathbf{x} \in K$. This means that $\mathbf{v} \in \mathrm{N}_K(\mathbf{k}')$. Hence, $\mathrm{N}_K(\mathbf{k}')$ is closed by Theorem 2.6.3. ■

Mazur's Finite Dimensional Density Theorem. *Smooth points are dense in the boundary of a convex body $K \subseteq \mathbb{E}^n$.*

The proof of Mazur's finite dimensional density theorem, above, relies on the well-known theorem below. Let $\{X_i\}$ be a countable collection of closed sets of the space $(\mathcal{X}, \hat{\mathrm{d}})$, which each have empty interior. The space $(\mathcal{X}, \hat{\mathrm{d}})$ is said to be a *Baire space* if $\bigcup X_i$ also has empty interior in $(\mathcal{X}, \hat{\mathrm{d}})$.

Baire Category Theorem. *If $(\mathcal{X}, \hat{\mathrm{d}})$ is a compact Hausdorff space or a complete metric space, then $(\mathcal{X}, \hat{\mathrm{d}})$ is a Baire space.*

A proof of the Baire Category Theorem can be found on page 296 of [42]. A topological space X is called a *Hausdorff space* if for each pair of distinct elements x_1 and x_2, there exists disjoint open sets U_1 and U_2 in X, which contain x_1 and x_2, respectively.

Lemma 3.2.2. *A subset of a Hausdorff space equipped with the subspace topology is a Hausdorff space.*

Proof. Let S be a subset of a Hausdorff space X and let s_1 and s_2 be arbitrarily chosen distinct elements from S. Since X is Hausdorff there exists two disjoint open sets U_1 and U_2 such that $s_1 \in U_1$ and $s_2 \in U_2$. Then, $s_1 \in U_1 \cap S$ and $s_2 \in U_2 \cap S$ where $U_1 \cap S, U_2 \cap S \subseteq \mathcal{T}_S = \{S \cap U \mid U \text{ is open in} X\}$, which implies that $U_1 \cap S$ and $U_2 \cap S$ are open in $(S, \mathcal{T}_S)$. Moreover,

$$(U_1 \cap S) \cap (U_2 \cap S) = (U_1 \cap U_2) \cap S = \varnothing \cap S = \varnothing.$$

Hence, $(S, \mathcal{T}_S)$ is Hausdorff. ■

The proof of the above lemma, used in the proof of Mazur's Finite Dimensional Density Theorem, is due to [41]. All metric spaces are Hausdorff spaces (see p. 129 of [42]). Therefore, $\mathbb{E}^n$ is a Hausdorff space.

Theorem 3.2.3. *Let $S \subseteq \mathbb{E}^n$. Then, $(S, \mathcal{T}_S)$ is compact if and only if it is closed and bounded in the Euclidean metric.*

A proof of the above theorem can be found on p. 173 of [42]. The overall structure of the proof of Mazur's Finite Dimensional Density Theorem is due to [26] and the proof of Claim 1 is due to [48].

Proof of Mazur's Finite Dimensional Density Theorem.
Let $K \subseteq \mathbb{E}^n$ be an arbitrarily chosen convex body. First, it will be shown that $\text{bd}(K)$ with the subspace topology is a Baire space.

Since K is compact in $\mathbb{E}^n$, it follows that K is closed and bounded. By Theorem 2.5.7 and Theorem 2.5.1, $K = \text{bd}(K) \cup \text{int}(K)$ where $\text{bd}(K) \cap \text{int}(K) = \varnothing$. This means that $\text{bd}(K) \subseteq K$. This together with $\mathbb{E}^n$ being Hausdorff and Lemma 3.2.2 imply that $\left(\text{bd}(K), \mathcal{T}_{\text{bd}(K)}\right)$ is a Hausdorff space.

By definition, $\mathrm{bd}(K) = \mathrm{cl}(\mathbb{E}^n \backslash K) \cap \mathrm{cl}(K)$. In particular, this implies that $\mathrm{bd}(K)$ is closed in $\mathbb{E}^n$. Let $\mathbf{k}_1, \mathbf{k}_2 \in \mathrm{bd}(K)$ be arbitrarily chosen. Since $\mathrm{bd}(K) \subseteq K$, $\mathbf{k}_1, \mathbf{k}_2 \in K$. Recall that K is bounded in $\mathbb{E}^n$. Therefore, there exists a real number $M \geqslant 0$ such that $\|\mathbf{k}_1 - \mathbf{k}_2\| \leqslant M$. Thus, $\mathrm{bd}(K)$ is bounded in $\mathbb{E}^n$. By Theorem 3.2.3, $\left(\mathrm{bd}(K), \mathcal{T}_{\mathrm{bd}(K)}\right)$ is compact.
It follows from Baire Category Theorem that $\left(\mathrm{bd}(K), \mathcal{T}_{\mathrm{bd}(K)}\right)$ is a Baire space.

Now, it will be shown that the set of all singularities in $\mathrm{bd}(K)$ is meagre.
For all $m \in \{1, 2, \ldots\}$, let

$$S_m = \left\{\mathbf{k} \in \mathrm{bd}(K) \mid \exists\ \mathbf{u}, \mathbf{v} \in \mathrm{N}_K(\mathbf{k}) \cap \mathcal{S}^{n-1} \text{ such that } |\langle \mathbf{u}, \mathbf{v}\rangle| \leqslant 1 - \frac{1}{m}\right\}.$$

This means that for any $\mathbf{x} \in S_m$, there exists a distinct pair of outwards normal unit vectors from the normal cone of K at $\mathbf{x}$ whose angle is at least $\arccos\left(1 - {}^{1}\!/_{m}\right)$ and at most $\arccos\left({}^{1}\!/_{m} - 1\right)$, since the inverse cosine function is monotonically decreasing. It is clear that

$$\bigcup_{m=1}^{\infty} S_m = \{\mathbf{k} \in \mathrm{bd}(K) \mid \mathbf{k} \text{ is a singular point}\}.$$

Claim 1: Each S_m is closed in $\left(\mathrm{bd}(K), \mathcal{T}_{\mathrm{bd}(K)}\right)$, for all $m \in \{1, 2, \ldots\}$.
Let $\{\mathbf{x}_i\}_{i \in \mathbb{N}}$ be a convergent sequence whose elements belong to S_m. Denote the point to which $\{\mathbf{x}_i\}_{i \in \mathbb{N}}$ converges by $\mathbf{x}$. Since $S_m \subseteq \mathrm{bd}(K)$ and $\mathrm{bd}(K)$ is closed in $\mathbb{E}^n$, $\mathbf{x} \in \mathrm{bd}(K)$ by Theorem 2.6.3. For each element $\mathbf{x}_i \in S_m$ from the sequence $\{\mathbf{x}_i\}_{i \in \mathbb{N}}$, there exists $\mathbf{u}_i, \mathbf{v}_i \in \mathrm{N}_K(\mathbf{x}_i) \cap S^{n-1}$ such that

$$|\langle \mathbf{u}_i, \mathbf{v}_i\rangle| \leqslant 1 - \frac{1}{m}.$$

This creates two sequences of unit vectors $\{\mathbf{u}_i\}_{i \in \mathbb{N}}$ and $\{\mathbf{v}_i\}_{i \in \mathbb{N}}$ whose elements belong to

$$\bigcup_{i=1}^{\infty} \left(\mathrm{N}_K(\mathbf{x}_i) \cap \mathcal{S}^{n-1}\right) = S^{n-1} \cap \left(\bigcup_{i=1}^{\infty} \mathrm{N}_K(\mathbf{x}_i)\right) \subseteq S^{n-1}.$$

Let $\mathbf{s}_1, \mathbf{s}_2 \in S^{n-1}$ be arbitrarily chosen. It follows that $\|\mathbf{s}_1\| = \|\mathbf{s}_2\| = 1$. Use this together with the triangle inequality to get that

$$\|\mathbf{s}_1 - \mathbf{s}_2\| \leqslant \|\mathbf{s}_1\| + \|-\mathbf{s}_2\| = \|\mathbf{s}_1\| + |-1|\|\mathbf{s}_2\| = 2.$$

This means that S^{n-1} is bounded. By Theorem 2.7.1, the sequences $\{\mathbf{u}_i\}_{i\in\mathbb{N}}$ and $\{\mathbf{v}_i\}_{i\in\mathbb{N}}$ have convergent subsequences; denote the convergent subsequences by $\{\mathbf{u}_{i_j}\}_{i_j\in\mathbb{N}}$ and $\{\mathbf{v}_{i_j}\}_{i_j\in\mathbb{N}}$ and denote points to which they converge by $\mathbf{u}$ and $\mathbf{v}$ respectively. Note that S^{n-1} is closed by Theorem 2.5.1 and therefore, $\mathbf{u}, \mathbf{v} \in S^{n-1}$ by Theorem 2.6.3.

Now, it will be shown that $\mathbf{u}, \mathbf{v} \in \mathrm{N}_K(\mathbf{x})$. Let $\mathbf{k} \in K$ be arbitrarily chosen. To see that $\mathbf{u} \in \mathrm{N}_K(\mathbf{x})$, observe that

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{k}-\mathbf{x} \rangle &= \left\langle \mathbf{u}-\mathbf{u}_{i_j}, \mathbf{k}-\mathbf{x} \right\rangle + \left\langle \mathbf{u}_{i_j}, \mathbf{k}-\mathbf{x} \right\rangle \\
&\leqslant \left| \left\langle \mathbf{u}-\mathbf{u}_{i_j}, \mathbf{k}-\mathbf{x} \right\rangle \right| + \left\langle \mathbf{u}_{i_j}, \mathbf{k}-\mathbf{x} \right\rangle \\
&= \left| \left\langle \mathbf{u}-\mathbf{u}_{i_j}, \mathbf{k}-\mathbf{x} \right\rangle \right| + \left\langle \mathbf{u}_{i_j}, \mathbf{k}-\mathbf{x}_{i_j} \right\rangle + \left\langle \mathbf{u}_{i_j}, \mathbf{x}_{i_j}-\mathbf{x} \right\rangle \\
&\leqslant \left| \left\langle \mathbf{u}-\mathbf{u}_{i_j}, \mathbf{k}-\mathbf{x} \right\rangle \right| + \left\langle \mathbf{u}_{i_j}, \mathbf{k}-\mathbf{x}_{i_j} \right\rangle + \left| \left\langle \mathbf{u}_{i_j}, \mathbf{x}_{i_j}-\mathbf{x} \right\rangle \right|,
\end{aligned}$$

then, by the Cauchy Schwarz inequality,

$$\leqslant \|\mathbf{u}-\mathbf{u}_{i_j}\| \|\mathbf{k}-\mathbf{x}\| + \left\langle \mathbf{u}_{i_j}, \mathbf{k}-\mathbf{x}_{i_j} \right\rangle + \|\mathbf{u}_{i_j}\| \|\mathbf{x}_{i_j}-\mathbf{x}\|,$$

then, $\left\langle \mathbf{u}_{i_j}, \mathbf{k} \right\rangle \leqslant \left\langle \mathbf{u}_{i_j}, \mathbf{x}_{i_j} \right\rangle$ since $\mathbf{u}_{i_j} \in \mathrm{N}_K\left(\mathbf{x}_{i_j}\right) \cap S^{n-1}$ and thus,

$$\leqslant \|\mathbf{u}-\mathbf{u}_{i_j}\| \|\mathbf{k}-\mathbf{x}\| + \|\mathbf{u}_{i_j}\| \|\mathbf{x}_{i_j}-\mathbf{x}\|,$$

then, since $\|\mathbf{u}_{i_j}\| = 1$ because $\mathbf{u}_{i_j} \in \mathrm{N}_K\left(\mathbf{x}_{i_j}\right) \cap S^{n-1}$,

$$= \|\mathbf{u}-\mathbf{u}_{i_j}\| \|\mathbf{k}-\mathbf{x}\| + \|\mathbf{x}_{i_j}-\mathbf{x}\|. \tag{$\star$}$$

If $\mathbf{k} = \mathbf{x}$, then

$$(\star) = \|\mathbf{x}_{i_j}-\mathbf{x}\| < \varepsilon,$$

since the sequence $\{\mathbf{x}_i\}_{i\in\mathbb{N}}$ converges to $\mathbf{x}$ and therefore, the subsequence $\{\mathbf{x}_{i_j}\}_{i_j\in\mathbb{N}}$ converges to $\mathbf{x}$ by Theorem 2.6.1. Recall, from above, that the distance between any two elements of K is less than or equal to some $M \in \mathbb{R}$. Suppose $\mathbf{k} \neq \mathbf{x}$ and $M \geqslant 1$. Since $\mathbf{u}_{i_j} \to \mathbf{u}$ and $\mathbf{x}_{i_j} \to \mathbf{x}$ as $i_j \to \infty$, there exists $N_1 \in \mathbb{N}$ such that for any $i_j \geqslant N_1$,

$$\|\mathbf{u}_{i_j}-\mathbf{u}\| < \frac{\varepsilon}{2M}$$

and there exists $N_2 \in \mathbb{N}$ such that for any $i_j \geqslant N_2$,

$$\|\mathbf{x}_{i_j} - \mathbf{x}\| < \frac{\varepsilon}{2M}.$$

Thus, for any $i_j \geqslant N$ where $N = \max\{N_1, N_2\}$,

$$(\star) \leqslant M\|\mathbf{u}_{i_j} - \mathbf{u}\| + M\|\mathbf{x}_{i_j} - \mathbf{x}\| < M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon.$$

Now, suppose that $\mathbf{k} \neq \mathbf{x}$ and $M < 1$. Again, since $\mathbf{u}_{i_j} \to \mathbf{u}$ and $\mathbf{x}_{i_j} \to \mathbf{x}$ as $i_j \to \infty$, there exists $N_1' \in \mathbb{N}$ such that for any $i_j \geqslant N_1'$,

$$\|\mathbf{u}_{i_j} - \mathbf{u}\| < \frac{\varepsilon}{2}$$

and there exists $N_2' \in \mathbb{N}$ such that for any $i_j \geqslant N_2'$,

$$\|\mathbf{x}_{i_j} - \mathbf{x}\| < \frac{\varepsilon}{2}.$$

Thus, for any $i_j \geqslant N'$ where $N' = \max\{N_1', N_2'\}$,

$$(\star) < \|\mathbf{u}_{i_j} - \mathbf{u}\| + \|\mathbf{x}_{i_j} - \mathbf{x}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

As $\varepsilon \to 0$, $\langle \mathbf{u}, \mathbf{k} - \mathbf{x} \rangle \leqslant 0$. This means that $\langle \mathbf{u}, \mathbf{k} \rangle \leqslant \langle \mathbf{u}, \mathbf{x} \rangle$ for any $\mathbf{k} \in K$. Hence, $\mathbf{u} \in \mathrm{N}_K(\mathbf{x})$. A nearly identical argument can be used to show that $\mathbf{v} \in \mathrm{N}_K(\mathbf{x})$. Thus, $\mathbf{u}, \mathbf{v} \in \mathrm{N}_K(\mathbf{x}) \cap S^{n-1}$.

The following argument will show that $|\langle \mathbf{u}, \mathbf{v} \rangle| \leqslant 1 - \frac{1}{m}$. First, notice that the sequence of real numbers $\{|\langle \mathbf{u}_{i_j}, \mathbf{v}_{i_j} \rangle|\}_{i_j \in \mathbb{N}}$ converges to the real number $|\langle \mathbf{u}, \mathbf{v} \rangle|$. To see this, observe that by the reverse triangle inequality

$$\begin{aligned} \Big| |\langle \mathbf{u}_{i_j}, \mathbf{v}_{i_j} \rangle| - |\langle \mathbf{u}, \mathbf{v} \rangle| \Big| &\leqslant |\langle \mathbf{u}_{i_j}, \mathbf{v}_{i_j} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle| \\ &= |\langle \mathbf{u}_{i_j}, \mathbf{v}_{i_j} \rangle + \langle \mathbf{u}_{i_j}, -\mathbf{v} \rangle + \langle \mathbf{u}_{i_j}, \mathbf{v} \rangle + \langle -\mathbf{u}, \mathbf{v} \rangle| \\ &= |\langle \mathbf{u}_{i_j}, \mathbf{v}_{i_j} - \mathbf{v} \rangle + \langle \mathbf{u}_{i_j} - \mathbf{u}, \mathbf{v} \rangle|, \end{aligned}$$

then, by the triangle inequality,

$$\leqslant |\langle \mathbf{u}_{i_j}, \mathbf{v}_{i_j} - \mathbf{v} \rangle| + |\langle \mathbf{u}_{i_j} - \mathbf{u}, \mathbf{v} \rangle|,$$

then, by the Cauchy Schwarz inequality,

$$
\begin{aligned}
&\leqslant \|\mathbf{u}_{i_j}\|\|\mathbf{v}_{i_j}-\mathbf{v}\| + \|\mathbf{u}_{i_j}-\mathbf{u}\|\|\mathbf{v}\| \\
&= \|\mathbf{v}_{i_j}-\mathbf{v}\| + \|\mathbf{u}_{i_j}-\mathbf{u}\| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

since $\mathbf{u}_{i_j} \to \mathbf{u}$ and $\mathbf{v}_{i_j} \to \mathbf{v}$. Recall that $|\langle \mathbf{u}_i, \mathbf{v}_i \rangle| \leqslant 1 - \dfrac{1}{m}$ for each $i \in \mathbb{N}$ and thus, $\left|\left\langle \mathbf{u}_{i_j}, \mathbf{v}_{i_j} \right\rangle\right| \leqslant 1 - \dfrac{1}{m}$ for each $i_j \in \mathbb{N}$. Therefore, it follows from Theorem 2.6.4 that $|\langle \mathbf{u}, \mathbf{v} \rangle| \leqslant 1 - \dfrac{1}{m}$.

Hence, $\mathbf{x} \in S_m$. In particular, this implies that S_m is closed in $\mathbb{E}^n$ by Theorem 2.6.3. It follows from Theorem 2.5.2 that S_m is closed in $\left(\mathrm{bd}(K), \mathcal{T}_{\mathrm{bd}(K)}\right)$.

Claim 2: Each S_m is nowhere dense in $\left(\mathrm{bd}(K), \mathcal{T}_{\mathrm{bd}(K)}\right)$.

Suppose, for a contradiction, that S_m is not nowhere dense.

It follows that $\mathrm{int}_{\mathrm{bd}(K)}\left(\mathrm{cl}_{\mathrm{bd}(K)}\left(S_m\right)\right) \neq \varnothing$. Recall from Claim 1 that S_m is closed in $\left(\mathrm{bd}(K), \mathcal{T}_{\mathrm{bd}(K)}\right)$ and therefore, $\mathrm{int}_{\mathrm{bd}(K)}\left(S_m\right) \neq \varnothing$. Let $\mathbf{x} \in \mathrm{int}_{\mathrm{bd}(K)}\left(S_m\right)$ be arbitrarily chosen. It follows that there exists an open set U in $\left(\mathrm{bd}(K), \mathcal{T}_{\mathrm{bd}(K)}\right)$ such that $\mathbf{x} \in U$ and $U \subseteq S_m$. Note that

$$
\mathrm{bd}(U) \cap U = \mathrm{bd}_{\mathrm{bd}(K)}\left(U\right) \cap \mathrm{int}_{\mathrm{bd}(K)}\left(U\right) = \varnothing,
$$

by Theorem 2.5.1 and Theorem 2.5.7.

Suppose $U \subseteq \mathrm{conv}\left(\mathrm{bd}(U)\right)$. This means that for any $\mathbf{u} \in U$, there exists $\mathbf{u}_1, \mathbf{u}_2 \in \mathrm{bd}(U)$ and $0 < \lambda < 1$ such that $\mathbf{u} = \lambda \mathbf{u}_1 + (1-\lambda)\mathbf{u}_2$.

Suppose $U \nsubseteq \mathrm{conv}\left(\mathrm{bd}(U)\right)$. This means that there exists $\mathbf{z} \in U$ such that

$$
\mathbf{z} \notin \mathrm{conv}\left(\mathrm{bd}(U)\right).
$$

By Theorem 2.10.1.5, there exists a hyperplane, H, which properly separates $\{\mathbf{z}\}$ from $\mathrm{conv}\left(\mathrm{bd}(U)\right)$. Suppose without loss of generality that $\mathbf{z} \in \overline{H^+}$. Note that $K \cap \overline{H^+}$ is a convex body. Let B be the maximum volume ball contained in the convex body $K \cap \overline{H^+}$. It follows that B must touch the boundary of $K \cap H^+ \subseteq U$; denote the point at which this occurs by $\mathbf{b}$. Clearly, there is a unique supporting hyperplane of K at $\mathbf{b} \in S_m$, which is a contradiction.

The set of all smooth points can be expressed as

$$\mathrm{bd}(K) \backslash \bigcup_{m=1}^{\infty} S_m .$$

By Theorem 2.5.4 and Claim 2,

$$\mathrm{cl}_{\mathrm{bd}(K)} \left(\mathrm{bd}(K) \backslash \bigcup_{m=1}^{\infty} S_m \right) = \mathrm{bd}(K) \backslash \mathrm{int}_{\mathrm{bd}(K)} \left(\bigcup_{m=1}^{\infty} S_m \right) = \mathrm{bd}(K) \backslash \varnothing = \mathrm{bd}(K).$$

Hence, $\mathrm{bd}(K) \backslash \bigcup_{m=1}^{\infty} S_m$ is dense in $\big(\mathrm{bd}(K), \mathcal{T}_{\mathrm{bd}(K)}\big)$.

■

3.3 The John - Löwner Theorem

An *ellipsoid*, $\mathcal{E}$, in $\mathbb{E}^n$ with centre $\mathbf{a}$ is defined to be

$$\mathcal{E} = T(\mathrm{B}^n [\mathbf{o}, 1]) + \mathbf{a}$$

where $\mathrm{B}^n [\mathbf{o}, 1] = \{\mathbf{x} \in \mathbb{E}^n \mid \|\mathbf{x}\| \leqslant 1\}$ is the closed Euclidean unit ball, $\mathbf{a} \in \mathbb{E}^n$ is a vector and $T : \mathbb{E}^n \to \mathbb{E}^n$ is an invertible linear transformation.

Let A be an $n \times n$ matrix. Denote the entry in the i-th row and j-th column of A by a_{ij}. Recall that a matrix A is *symmetric* if $a_{ij} = a_{ji}$, for all $1 \leqslant i, j \leqslant n$. A symmetric matrix A is *positive definite* if $\langle \mathbf{x}, A\mathbf{x} \rangle > 0$, for all vectors $\mathbf{x} \neq \mathbf{o}$.

The proof of the result below follows from a comment on page 86 and Example 3 on page 81 of [28].

Proposition 3.3.1. *Let A be an $n \times n$ matrix and let $\sigma_j = (j_1, j_2, \ldots, j_n)$ denote one element from the set of all $n!$ permutations of the integers from 1 to n, which is labelled by S_n. The permutation σ_j is said to be even if an even number of two element exchanges are required when starting from the natural ordering $(1, 2, \ldots, i, i+1, \ldots, n-1, n)$ to get it into the form*

$(j_1, j_2, \ldots, j_n)$. *Likewise, the permutation is said to be odd if an odd number of two element exchanges are required. For example, the permutation* $(3,2,1)$ *is odd since it is obtained from exchanging the numbers* 1 *and* 3 *in the permutation* $(1,2,3)$. *If* σ_j *is even, then* $\operatorname{sign}(\sigma_j) = 1$ *and if* σ_j *is odd, then* $\operatorname{sign}(\sigma_j) = -1$. *The determinant of* A *is denoted and defined by*

$$\det(A) = \sum_{\sigma_j \in S_n} \operatorname{sign}(\sigma_j) \prod_{i=1}^{n} a_{ij_i}.$$

The determinant map, which sends $\mathbb{E}^{n\times n}$ *to* $\mathbb{R}$, *is a continuous function.*

The properties of the determinant listed below are useful; their proofs can be found after Theorem 2 on page 49 of [33], Theorem 2 on page 117 of [43] and Theorem 5 on page 113 of [43], respectively.

Properties 3.3.2. *For any* $n \times n$ *matrices* A *and* B,

(i) $\det(AB) = \det(A)\det(B)$;

(ii) A *is invertible if and only if* $\det(A) \neq 0$;

(iii) *If* A *is invertible, then* $\det(A^{-1}) = \dfrac{1}{\det(A)}$;

(iv) *Let* A *and* B *be square matrices and let*

$$C = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$$

be a block matrix where 0 *represents a matrix whose entries are all zeros. Then,* $\det(C) = \det(A)\det(B)$.

Below, is a collection of properties of positive definite matrices; their proofs follow from Theorem 1 on page 144, Theorem 3 on page 65 and Corollary 3 on page 51 of [33].

Properties 3.3.3. *Let* A *be a symmetric positive definite* $n \times n$ *matrix. Then,*

(i) *Each eigenvalue* λ_i *of* A *is positive, for* $1 \leqslant i \leqslant n$. *Also, if each eigenvalue of some symmetric* $n \times n$ *matrix* B *is positive, then* B *is positive definite;*

(ii) $\det(A) > 0$;

(iii) A *is invertible.*

The matrix resulting from interchanging the rows and columns of A and applying the complex conjugate to each entry in the matrix A is called the *conjugate transpose* of A and is denoted by A^*. It is stated on on page 131 of [56] that $\det(A^*) = \overline{\det(A)}$, where $\overline{\det(A)}$ denotes the complex conjugate of $\det(A)$.

Proposition 3.3.4. *Any ellipsoid $\mathcal{E}$ with centre $\mathbf{a}$ can be written as*

$$\mathcal{E} = \{\mathbf{x} \in \mathbb{E}^n \mid \langle Q(\mathbf{x} - \mathbf{a}), \mathbf{x} - \mathbf{a} \rangle \leqslant 1\}$$

for some positive definite matrix Q.

Proof. By definition $\mathcal{E} = T(\mathrm{B}^n[\mathbf{o}, 1]) + \mathbf{a}$ for some invertible linear transformation $T : \mathbb{E}^n \to \mathbb{E}^n$ and some vector $\mathbf{a} \in \mathbb{E}^n$. This means

$$\begin{aligned}
\mathcal{E} = T(\mathrm{B}^n[\mathbf{o}, 1]) + \mathbf{a} &= \{T(\mathbf{x}) + \mathbf{a} \mid \mathbf{x} \in \mathrm{B}^n[\mathbf{o}, 1]\} \\
&= \{T(\mathbf{x}) + \mathbf{a} \mid \langle \mathbf{x}, \mathbf{x} \rangle \leqslant 1\} \\
&= \{\mathbf{x} \in \mathbb{E}^n \mid \left\langle T^{-1}(\mathbf{x} - \mathbf{a}), T^{-1}(\mathbf{x} - \mathbf{a}) \right\rangle \leqslant 1\} \\
&= \{\mathbf{x} \in \mathbb{E}^n \mid \left\langle T^{*-1} T^{-1}(\mathbf{x} - \mathbf{a}), \mathbf{x} - \mathbf{a} \right\rangle \leqslant 1\} \\
&= \{\mathbf{x} \in \mathbb{E}^n \mid \left\langle (TT^*)^{-1}(\mathbf{x} - \mathbf{a}), \mathbf{x} - \mathbf{a} \right\rangle \leqslant 1\}
\end{aligned}$$

Let $A \in \mathbb{R}^{n \times n}$ be the matrix representation of the linear transformation T. Since T is invertible, $A^{-1} \in \mathbb{R}^{n \times n}$ is the matrix representation of the transformation $T^{-1} : \mathbb{E}^n \to \mathbb{E}^n$. Denote the matrix $(AA^*)^{-1}$ by Q. First notice that Q is self-adjoint. Namely,

$$\begin{aligned}
Q^* = \left((AA^*)^{-1}\right)^* &= \left(A^{-1*} A^{-1}\right)^* \\
&= A^{-1*} A^{-1} \\
&= A^{*-1} A^{-1} = (AA^*)^{-1} = Q.
\end{aligned}$$

For all $\mathbf{z} \neq \mathbf{o}$,

$$\begin{aligned}\langle Q\mathbf{z}, \mathbf{z}\rangle = \left\langle (AA^*)^{-1}\mathbf{z}, \mathbf{z}\right\rangle &= \left\langle A^{*\,-1}A^{-1}\mathbf{z}, \mathbf{z}\right\rangle \\ &= \left\langle A^{-1\,*}A^{-1}\mathbf{z}, \mathbf{z}\right\rangle \\ &= \left\langle A^{-1}\mathbf{z}, A^{-1}\mathbf{z}\right\rangle = \|A^{-1}\mathbf{z}\|^2 \geqslant 0.\end{aligned}$$

In fact, $\langle Q\mathbf{z}, \mathbf{z}\rangle = \|A^{-1}\mathbf{z}\|^2 > 0$ since $A^{-1}\mathbf{z} \neq \mathbf{o}$. This follows from A^{-1} being invertible. Therefore, the matrix Q is positive definite and $\mathcal{E} = \{\mathbf{x} \in \mathbb{E}^n \mid \langle Q(\mathbf{x}-\mathbf{a}), \mathbf{x}-\mathbf{a}\rangle \leqslant 1\}$. ■

The *n-dimensional volume* of a set $S \subseteq \mathbb{E}^n$ is denoted as and defined by

$$\operatorname{vol}(S) = \int_S \cdots \int dx_1 \ldots dx_n.$$

For a proof of following result, see Theorem 14.15 on page 520 of [20].

Theorem 3.3.5. *Let $S : \mathbb{E}^n \to \mathbb{E}^n$ be a linear transformation and let B be a bounded subset of $\mathbb{E}^n$. Then,* $\operatorname{vol}(S(B)) = |\det(S)| \operatorname{vol}(B)$.

It follows from Proposition 3.3.4 and Properties 3.3.2 that

$$\det(Q) = \frac{1}{\det(T)^2}.$$

This together with the above theorem implies that the volume of any ellipsoid

$$\mathcal{E} = T\left(\mathrm{B}^n\left[\mathbf{o}, 1\right]\right) + \mathbf{a}$$

is

$$\operatorname{vol}(\mathcal{E}) = |\det(T)| \operatorname{vol}\left(\mathrm{B}^n\left[\mathbf{o}, 1\right]\right) = \frac{\operatorname{vol}\left(\mathrm{B}^n\left[\mathbf{o}, 1\right]\right)}{\sqrt{\det(Q)}}.$$

Proofs of the next two facts can be found, respectively, after Lemma 1.2 and Lemma 1.3 on page 205 of [7].

Lemma 3.3.6. *Any ellipsoid $\mathcal{E}$ can be written as $S\left(\mathrm{B}^n\left[\mathbf{o}, 1\right]\right) + \mathbf{a}$ where $\mathbf{a} \in \mathbb{E}^n$ and $S : \mathbb{E}^n \to \mathbb{E}^n$ is a linear transformation which is induced by a positive definite matrix.*

Lemma 3.3.7. *Let Q_1 and Q_2 be $n \times n$ positive definite matrices. Then,*

$$\det\left(\frac{1}{2}(Q_1 + Q_2)\right) \geqslant \sqrt{\det(Q_1)\det(Q_2)}.$$

Equality holds if and only if $Q_1 = Q_2$.

A function $T : \mathbb{E}^n \to \mathbb{E}^n$ is an *orthogonal transformation* if $\langle T(\mathbf{x}), T(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{E}^n$. The set of non-zero vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ in $\mathbb{E}^n$ is called an *orthonormal set* if $\|\mathbf{x}_i\| = 1$, for all $1 \leqslant i \leqslant m$ and $\langle \mathbf{x}_i, \mathbf{x}_j\rangle = 0$, for all $i \neq j$. The following fact provides a further characterization of orthogonal transformations; it is proved on pages 16 and 17 of [49].

Proposition 3.3.8. *If $T : \mathbb{E}^n \to \mathbb{E}^n$ is an orthogonal transformation, then T is linear and $\{T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)\}$ is an orthonormal basis of $\mathbb{E}^n$, where $\mathbf{e}_1, \ldots \mathbf{e}_n$ denote the standard basis vectors of $\mathbb{E}^n$.*

The collection of properties of orthogonal transformations, below, rely on the previous result. Their proofs can be found on page 17 of [49], page 161 of [38], page 119 of [43] and page 328 of [43].

Corollary 3.3.9. *Let $T : \mathbb{E}^n \to \mathbb{E}^n$ be an orthogonal transformation induced by the matrix $P = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n)\end{bmatrix}$. Note that P is called an orthogonal matrix. Then,*

(i) $\|T(\mathbf{x}) - T(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$, *for any $\mathbf{x}, \mathbf{y} \in \mathbb{E}^n$;*

(ii) *$PP^T = I_n = P^TP$ implying that $P^{-1} = P^T$;*

(iii) *either $\det(P) = 1$ or $\det(P) = -1$;*

(iv) *the rows and columns of P are an orthonormal basis for $\mathbb{E}^n$.*

The following well-known theorem, describes how to find an orthogonal basis of a subspace $S \subseteq \mathbb{E}^n$ from any basis of S; a proof of the Gram-Schmidt Orthogonalization Theorem can be found on pages 51 and 52 of [47].

Gram-Schmidt Orthogonalization Theorem. *Let* $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ *be a basis of a subspace* $S \subseteq \mathbb{E}^n$. *Then, the set containing the vectors*

$$\begin{aligned}
\mathbf{f}_1 &= \mathbf{x}_1 \\
\mathbf{f}_2 &= \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{f}_1 \rangle}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 \\
&\vdots \\
\mathbf{f}_m &= \mathbf{x}_m - \frac{\langle \mathbf{x}_m, \mathbf{f}_1 \rangle}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\langle \mathbf{x}_m, \mathbf{f}_2 \rangle}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 - \ldots - \frac{\langle \mathbf{x}_m, \mathbf{f}_{m-1} \rangle}{\|\mathbf{f}_{m-1}\|^2}\mathbf{f}_{m-1}
\end{aligned}$$

is an orthogonal basis of S *and the set of all linear combinations of* $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ *is equal to the set of all linear combinations of* $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$.

The theorem below characterizes the bases of $\mathbb{E}^n$; its proof can be found on pages 46 and 47 of [5].

Theorem 3.3.10. *Any set of* n *linearly independent vectors in* $\mathbb{E}^n$ *is a basis of* $\mathbb{E}^n$.

Two $n \times n$ matrices A and B are called *similar* if there exists some invertible matrix P such that $B = P^{-1}AP$. The lemma below describes some properties of similar matrices; its proof can be found on page 229 of [43].

Lemma 3.3.11. *If* A *and* B *are similar* $n \times n$ *matrices, then* A *and* B *have the same determinant and eigenvalues.*

An $n \times n$ matrix A is said to be *orthogonally diagonalizable* if there exists an orthogonal matrix P such that $P^T AP$ is a diagonal matrix. The following theorem characterizes which matrices have this property; its proof can be found on pages 329, 380 and 381 of [43].

Principal Axes Theorem. *Let* Q *be a symmetric* $n \times n$ *matrix and let*

$$R = \left\{\mathbf{x} \in \mathbb{E}^n \mid \mathbf{x}^T Q\mathbf{x}\right\} = \{\mathbf{x} \in \mathbb{E}^n \mid \langle \mathbf{x}, Q\mathbf{x} \rangle\}$$

be a quadratic form in the variables $x_1, x_2, \ldots x_n$. *Then,* Q *has an orthonormal set of eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *such that the orthogonal matrix*

$$P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_n \end{bmatrix}$$

orthogonally diagonalizes Q. The quadratic form in terms of the new variables $y_1, y_2, \ldots, y_n$ for $\mathbf{y} = P^T(\mathbf{x})$ is $R = \left\{\mathbf{y} \in \mathbb{E}^n \mid \left\langle \mathbf{y}, P^T AP(\mathbf{y})\right\rangle\right\} = \{\mathbf{y} \in \mathbb{E}^n \mid \lambda_1 y_1 + \lambda_2 y_2 + \ldots + \lambda_n y_n\}$, where $\lambda_1, \ldots \lambda_n$ are the eigenvalues of Q repeated according to their multiplicities. Note that the columns of P are called the principal axes of the quadratic form R.

The concept of length or norm of a vector developed in Chapter 2 can be extended to matrices. The *Frobenius norm* is a map which sends a matrix A of $\mathbb{E}^{m\times n}$ to the real number

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2}.$$

Concatenate the columns of the matrix $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \ddots & & \\ a_{m1} & a_{m_2} & \ldots & a_{mn} \end{bmatrix} \in E^{m\times n}$ into a single vector $\mathbf{A} = \begin{bmatrix} a_{11} & a_{21} & \ldots & a_{m1} & \ldots & a_{1n} & a_{2n} & \ldots & a_{mn} \end{bmatrix}^T \in \mathbb{E}^{mn}$. Notice that the Euclidean norm of the vector $\mathbf{A}$ is equivalent to the Frobenius norm of the matrix A:

$$\|\mathbf{A}\| = \sqrt{\langle \mathbf{A}, \mathbf{A}\rangle} = \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{m} (a_{ij})^2} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2} = \|A\|_F.$$

Proposition 3.3.12. *Let $A \in \mathbb{E}^{m\times n}$ and $\mathbf{x} \in \mathbb{E}^n$. Then,*

$$\|A\mathbf{x}\| \leqslant \|A\|_F \|\mathbf{x}\|$$

and

$$\|A\|_F \leqslant \sqrt{\min(m, n)} \cdot \max_{\|\mathbf{x}\|=1} \|A(\mathbf{x})\|.$$

The first property of the Frobenius norm in the above proposition is stated on page 280 of [40] and labelled as equation (5.2.5); the second is stated on page 133 of [23] and labelled as equation (3.238).

John - Löwner Theorem. *There exists a unique ellipsoid $\mathcal{E}$ of maximal volume contained in some convex body $K \subseteq \mathbb{E}^n$. Furthermore,*

$$\mathcal{E} \subseteq K \subseteq n\,\mathcal{E}.$$

Proof of the John - Löwner Theorem.

Let K be an arbitrarily chosen convex body. It follows from Lemma 3.3.6 that for each ellipsoid $\hat{\mathcal{E}} \subseteq K$ there exists some vector $\mathbf{a} \in \mathbb{E}^n$ and some linear transformation $S : \mathbb{E}^n \to \mathbb{E}^n$ which is induced by a positive definite matrix such that $\hat{\mathcal{E}} = S\left(\mathrm{B}^n\left[\mathbf{o}, 1\right]\right) + \mathbf{a}$. Let $\mathcal{X}$ denote the set of all pairs $(\mathbf{S}, \mathbf{a})$.

First, it will be shown that there exists a maximal volume ellipsoid inscribed in K.

Claim: $\mathcal{X}$ is compact in $\mathbb{E}^{n^2+n}$.

Let $(\mathbf{S}_1, \mathbf{a}_1), (\mathbf{S}_2, \mathbf{a}_2) \in \mathcal{X}$ be arbitrarily chosen. Then, $S_1(\mathbf{x}) + \mathbf{a}_1, S_2(\mathbf{x}) + \mathbf{a}_2 \in K$, for any $\mathbf{x} \in \mathrm{B}^n\left[\mathbf{o}, 1\right]$. Since K is compact and thus, bounded, there exists a real number M such that $\|S_1(\mathbf{x}) + \mathbf{a}_1\| \leqslant M$ and $\|S_2(\mathbf{x}) + \mathbf{a}_2\| \leqslant M$. For any $\mathbf{x} \in \mathrm{B}^n\left[\mathbf{o}, 1\right]$,

$$\|(S_1 - S_2)(\mathbf{x}) + (\mathbf{a}_1 - \mathbf{a}_2)\| = \|S_1(\mathbf{x}) + \mathbf{a}_1 - (S_2(\mathbf{x}) + \mathbf{a}_2)\|$$

then, by the triangle inequality and a property of the norm,

$$\begin{aligned} &\leqslant \|S_1(\mathbf{x}) + \mathbf{a}_1\| + |-1|\,\|S_2(\mathbf{x}) + \mathbf{a}_2\| \\ &\leqslant M + M = 2M. \end{aligned} \tag{$\star$}$$

Since linear transformations preserve the zero vector, it follows that

$$\mathbf{o} \in S_1\left(\mathrm{B}^n\left[\mathbf{o}, 1\right]\right), S_2\left(\mathrm{B}^n\left[\mathbf{o}, 1\right]\right).$$

Then, $\mathbf{a}_1 = \mathbf{o} + \mathbf{a}_1 \in S_1\left(\mathrm{B}^n\left[\mathbf{o}, 1\right]\right) + \mathbf{a}_1 \subseteq K$ and $\mathbf{a}_2 = \mathbf{o} + \mathbf{a}_2 \in S_2\left(\mathrm{B}^n\left[\mathbf{o}, 1\right]\right) + \mathbf{a}_2 \subseteq K$. Thus,

$$\|\mathbf{a}_1 - \mathbf{a}_2\| \leqslant M, \tag{$\dagger$}$$

since K is bounded. Notice that by Proposition 3.3.12,

$$\|S_1 - S_2\|_F \leqslant \sqrt{n} \cdot \max_{\|\mathbf{x}\|=1} \|(S_1 - S_2)(\mathbf{x})\|$$

then, by ($\star$),

$$\leqslant \sqrt{n} \cdot 2M = 2\sqrt{n}M, \tag{$\ddagger$}$$

for all $\mathbf{x} \in \mathrm{B}^n[\mathbf{o}, 1]$. Hence,

$$\begin{aligned} \|(\mathbf{S}_1, \mathbf{a}_1) - (\mathbf{S}_2, \mathbf{a}_2)\| &= \|(\mathbf{S}_1 - \mathbf{S}_2, \mathbf{a}_1 - \mathbf{a}_2)\| \\ &= \|(\mathbf{S}_1 - \mathbf{S}_2, \mathbf{o}) + (\underbrace{\mathbf{o}}_{\in \mathbb{E}^{n^2+n}}, \mathbf{a}_1 - \mathbf{a}_2)\| \end{aligned}$$

then, by the triangle inequality,

$$\leqslant \|(\mathbf{S}_1 - \mathbf{S}_2, \mathbf{o})\| + \|(\mathbf{o}, \mathbf{a}_1 - \mathbf{a}_2)\|$$

then, by ($\ddagger$) and ($\dagger$)

$$, \leqslant 2\sqrt{n}M + M = M\left(2\sqrt{n} + 1\right).$$

This means that $\mathcal{X} \subseteq \mathbb{E}^{n^2+n}$ is bounded.

Let $\{(\mathbf{S}_i, \mathbf{a}_i)\}_{i \in \mathbb{N}}$ be a convergent sequence whose elements belong to $\mathcal{X} \subseteq \mathbb{E}^{n^2+n}$; denote the point to which the sequence converges by $\mathbf{x} \in \mathbb{E}^{n^2+n}$. Let $X = \begin{bmatrix} x_1 & x_{n+1} & \dots & x_{n^2-n} \\ \vdots & \ddots & & \\ x_n & x_{2n} & \dots & x_{n^2} \end{bmatrix}$ and $\hat{\mathbf{x}} = \begin{bmatrix} x_{n^2+1} & \dots x_{n^2+n} \end{bmatrix}^T$. Consider the sequence $\{\mathbf{a}_i\}_{i \in \mathbb{N}}$. Suppose for a contradiction that $\{\mathbf{a}_i\}_{i \in \mathbb{N}}$ does not converge to $\hat{\mathbf{x}}$. This means that there exists $\varepsilon > 0$ such that $\|\mathbf{a}_i - \hat{\mathbf{x}}\| > \varepsilon$ for all $i \in \mathbb{N}$. However,

$$\varepsilon > \|(\mathbf{S}_i, \mathbf{a}_i) - \mathbf{x}\| = \|(\mathbf{S}_i - \mathbf{X}, \mathbf{o}) - (\mathbf{o}, \mathbf{a}_i - \hat{\mathbf{x}})\|$$

then, by the reverse triangle inequality,

$$\begin{aligned} &\geqslant \big| \|(\mathbf{S}_i - \mathbf{X}, \mathbf{o})\| - |(\mathbf{o}, \mathbf{a}_i - \hat{\mathbf{x}})\| \ \big| \\ &= \big|\|S_i - X\|_F - \|\mathbf{a}_i - \mathbf{x}\|\big| \end{aligned}$$

then, since $\|\mathbf{a}_i - \hat{\mathbf{x}}\| > \varepsilon$ and by a property of the Frobenius norm,

$$> |0 - \varepsilon| = \varepsilon,$$

for all $i \in \mathbb{N}$. This is a contradiction. Thus, the sequence $\{\mathbf{a}_i\}_{i\in\mathbb{N}}$ must converge to $\hat{\mathbf{x}}$. It follows that there exists some $N \in \mathbb{N}$ such that

$$\begin{aligned} \|S_i - X\|_F &= \|(\mathbf{S}_i - \mathbf{X}, \mathbf{o})\| \\ &= \|\,(\mathbf{S}_i - \mathbf{X}, \mathbf{o}) + (\mathbf{o}, \mathbf{a}_i - \hat{\mathbf{x}}) - (\mathbf{o}, \mathbf{a}_i - \hat{\mathbf{x}})\,\| \end{aligned}$$

then, by the triangle inequality and a property of the Euclidean norm,

$$\begin{aligned} &\leqslant \|(\mathbf{S}_i - \mathbf{X}, \mathbf{o}) + (\mathbf{o}, \mathbf{a}_i - \hat{\mathbf{x}})\| + |-1|\,\|(\mathbf{o}, \mathbf{a}_i - \hat{\mathbf{x}})\| \\ &= \|\,(\mathbf{S}_i, \mathbf{a}_i) - \mathbf{x}\| + \|\mathbf{a}_i - \hat{\mathbf{x}}\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

for all $i > N$. Thus, the sequence of linear transformations of $\mathbb{E}^n$ with positive semi-definite matrices $\{S_i\}_{i\in\mathbb{N}}$ converges to X. Let $\mathbf{z} \neq \mathbf{o}$ be arbitrarily chosen from $\mathrm{B}^n[\mathbf{o}, 1]$. Then, there exists $N' \in \mathbb{N}$ such that

$$|\langle \mathbf{z}, S_i(\mathbf{z})\rangle - \langle \mathbf{z}, X(\mathbf{z})\rangle| = |\langle \mathbf{z}, (S_i - X)(\mathbf{z})\rangle|$$

then, by the Cauchy-Schwarz inequality,

$$\leqslant \|\mathbf{z}\| \cdot \|\,(S_i - X)(\mathbf{z})\,\|$$

then, by Proposition 3.3.12,

$$\leqslant \|\mathbf{z}\|\|S_i - X\|_F\|\mathbf{z}\| = \|\mathbf{z}\|^2\,\|S_i - X\|_F$$

then, since $\{S_i\}_{i\in\mathbb{N}}$ converges to X,

$$< \|\mathbf{z}\|^2 \cdot \frac{\varepsilon}{\|\mathbf{z}\|^2} = \varepsilon,$$

for all $i > N'$. This means that the sequence of real numbers $\{\langle \mathbf{z}, S_i(\mathbf{z})\rangle\}_{i\in\mathbb{N}}$ converges to $\langle \mathbf{z}, X(\mathbf{z})\rangle$ for any $\mathbf{z} \neq \mathbf{o}$ from $\mathrm{B}^n[\mathbf{o},1]$. Since each S_i is positive definite, it follows from definition that each $\langle \mathbf{z}, S_i(\mathbf{z})\rangle > 0$ for any $\mathbf{z} \neq \mathbf{o}$ from $\mathrm{B}^n[\mathbf{o},1]$. Therefore, $\langle \mathbf{z}, X(\mathbf{z})\rangle > 0$ for any $\mathbf{z} \neq \mathbf{o}$ from $\mathrm{B}^n[\mathbf{o},1]$,by Theorem 2.6.4. Thus, X is a positive definite matrix.

It follows from Lemma 3.3.6 that $X(\mathrm{B}^n[\mathbf{o},1]) + \hat{\mathbf{x}}$ is an ellipse. However, it must be shown that $X(\mathrm{B}^n[\mathbf{o},1]) + \hat{\mathbf{x}} \subseteq K$. Let $\mathbf{z} \in X(\mathrm{B}^n[\mathbf{o},1]) + \hat{\mathbf{x}}$ be arbitrarily chosen. Then, there exists $\mathbf{b} \in X(\mathrm{B}^n[\mathbf{o},1])$ such that $\mathbf{z} = \mathbf{b} + b\hat{f}x$. Since X is positive definite, $\det(X) > 0$. Therefore, X is invertible. This means that $X^{-1}(\mathbf{b}) \in \mathrm{B}^n[\mathbf{o},1]$. Let $\mathbf{s}_i = S_i(X^{-1}(\mathbf{b}))$ for all $i \in \mathbb{N}$. Notice that there exists some $N'' \in \mathbb{N}$ such that

$$\|(\mathbf{s}_i + \mathbf{a}_i) - (\mathbf{b} - \hat{\mathbf{x}})\| = \|(\mathbf{s}_i + \mathbf{b}) - (\mathbf{a}_i - \hat{\mathbf{x}})\|$$

then, by the triangle inequality,

$$\begin{aligned} &\leqslant \|\mathbf{s}_i + \mathbf{b}\| + \|\mathbf{a}_i - \hat{\mathbf{x}}\| \\ &= \|S_i\left(X^{-1}(\mathbf{b})\right) - X\left(X^{-1}(\mathbf{b})\right)\| + \|\mathbf{a}_i - \hat{\mathbf{x}}\| \\ &= \|(S_i - X)\left(X^{-1}(\mathbf{b})\right)\| + \|\mathbf{a}_i - \hat{\mathbf{x}}\| \end{aligned}$$

then, by Proposition 3.3.12,

$$\leqslant \|S_i - X\|_F \|X^{-1}(\mathbf{b})\| + \|\mathbf{a}_i - \hat{\mathbf{x}}\|$$

then, since $X^{-1}(\mathbf{b}) \in \mathrm{B}^n[\mathbf{o},1]$,

$$\begin{aligned} &\leqslant \|S_i - X\|_F + \|\mathbf{a}_i - \hat{\mathbf{x}}\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

for all $i > N''$. This means that the sequence of points $\{\mathbf{s}_i + \mathbf{a}_i\}_{i\in\mathbb{N}}$ whose elements belong to K converges to $\mathbf{z}$. Since K is compact, it is also closed. Therefore, $\mathbf{z} \in K$ by Theorem 2.6.3. Thus, $X(\mathrm{B}^n[\mathbf{o},1]) + \hat{\mathbf{x}} \subseteq K$. Hence, $\mathcal{X}$ is closed in $\mathbb{E}^{n^2+n}$.

It follows that $\mathcal{X}$ is compact in $\mathbb{E}^{n^2+n}$. It follows from Proposition 3.3.1 that the function $f : \mathcal{X} \to \mathbb{R}$ with $f\big((\mathbf{S}, \mathbf{a})\big) = \det(S)$ is continuous. Therefore, by Extreme Value

Theorem $f : \mathcal{X} \to \mathbb{R}$ attains its maximum on $\mathcal{X}$, say at $(S', \mathbf{a}')$. Hence, the ellipsoid $\mathcal{E} = \det(S')\left(\mathrm{B}^n\left[\mathbf{o}, 1\right]\right) + \mathbf{a}'$ has maximum volume among all ellipsoids inscribed in K.

Now, it will be shown that the largest volume ellipsoid inscribed in K is unique. Suppose for a contradiction that this is not the case. In other words, suppose that the ellipsoids $\mathcal{E}_1 = S_1\left(\mathrm{B}^n\left[\mathbf{o}, 1\right]\right) + \mathbf{a}_1$ and $\mathcal{E}_2 = S_2\left(\mathrm{B}^n\left[\mathbf{o}, 1\right]\right) + \mathbf{a}_2$ are distinct, maximal volume ellipsoids contained in K. It follows that $\left(\mathbf{S}_1, \mathbf{a}_1\right), \left(\mathbf{S}_2, \mathbf{a}_2\right) \in \mathcal{X}$.

Claim: $\mathcal{X} \subseteq \mathbb{E}^{n^2+n}$ is convex.

Let $\left(\mathbf{S}, \mathbf{a}\right), \left(\mathbf{S}', \mathbf{a}'\right) \in \mathcal{X}$ and $\mathbf{x} \in \mathrm{B}^n\left[\mathbf{o}, 1\right]$ be arbitrarily chosen. It follows that $S\left(\mathbf{x}\right) + \mathbf{a} \in K$ and $S'\left(\mathbf{x}\right) + \mathbf{a}' \in K$. Now, let $0 \leqslant \lambda \leqslant 1$ be arbitrarily chosen. Then,

$$\lambda\left(S\left(\mathbf{x}\right) + \mathbf{a}\right) + (1 - \lambda)\left(S'\left(\mathbf{x}\right) + \mathbf{a}'\right) \in K,$$

since K is convex. Hence, $\lambda\left(\mathbf{S}, \mathbf{a}\right) + (1 - \lambda)\left(\mathbf{S}', \mathbf{a}'\right) \in \mathcal{X}$. This means that $\mathcal{X}$ is convex.

Let $S = \dfrac{1}{2}\left(S_1 + S_2\right)$ and $\mathbf{a} = \dfrac{1}{2}\left(\mathbf{a}_1 + \mathbf{a}_2\right)$. Since $\mathcal{X}$ is convex, $\left(\mathbf{S}, \mathbf{a}\right) \in \mathcal{X}$. It follows that the ellipsoid $S\left(\mathrm{B}^n\left[\mathbf{o}, 1\right]\right) + \mathbf{a}$ is contained in K; denote it by $\mathcal{E}'$. If $S_1 \neq S_2$, it follows from Lemma 3.3.7 that

$$\mathrm{vol}\left(\mathcal{E}'\right) = \det\left(S\right)\mathrm{vol}\left(\mathrm{B}^n\left[\mathbf{o}, 1\right]\right) > \mathrm{vol}\left(\mathcal{E}_1\right) = \mathrm{vol}\left(\mathcal{E}_2\right).$$

This would contradict the assumption that the ellipsoids $\mathcal{E}_1$ and $\mathcal{E}_2$ have the largest volume among all ellipsoids inscribed in K. Thus, $S_1 = S_2$.

In order for $\mathcal{E}_1$ and $\mathcal{E}_2$ to still be distinct, $\mathbf{a}_1 \neq \mathbf{a}_2$. Recall from the definition of an ellipsoid that S_1 is invertible. Recall from Lemma 2.4.6 that S_1^{-1} is a linear transformation. To make this part of the proof more tractable, apply the linear transformation S_1^{-1} to K. By *(ii)* of Properties 2.4.8, $S_1^{-1}\left(K\right)$ is still convex. It follows from Lemma 2.9.1 and Theorem 2.9.2 that $S_1^{-1}\left(K\right)$ is still compact and $S_1^{-1}\left(\mathrm{int}(K)\right)$ is non-empty since $\mathrm{int}(K) \neq \varnothing$ and S_1^{-1} is a function. In summary, $S_1^{-1}\left(K\right)$ is a convex body.

By Proposition 2.4.1, $S_1^{-1}\left(\mathcal{E}_1\right), S_1^{-1}\left(\mathcal{E}_2\right) \subseteq S_1^{-1}\left(K\right)$. Recall that for any ellipsoid $\mathcal{E} \subseteq K$,

$$\mathrm{vol}\left(\mathcal{E}\right) \leqslant \mathrm{vol}\left(\mathcal{E}_1\right) = \mathrm{vol}\left(\mathcal{E}_2\right). \qquad (✠)$$

It follows from Theorem 3.3.5 that

$$\operatorname{vol}\left(S_1^{-1}(\mathcal{E})\right) = \left|\det(S_1^{-1}\right| \operatorname{vol}(\mathcal{E})$$

then, by (✠) and the fact $\left|\det\left(S_1^{-1}\right)\right| > 0$,

$$\begin{aligned} &\leqslant \left|\det\left(S_1^{-1}\right)\right| \operatorname{vol}(\mathcal{E}_1) = \left|\det\left(S_1^{-1}\right)\right| \operatorname{vol}(\mathcal{E}_2) \\ &= \operatorname{vol}\left(S_1^{-1}(\mathcal{E}_1)\right) = \operatorname{vol}\left(S_1^{-1}(\mathcal{E}_2)\right), \end{aligned}$$

meaning that the ellipsoids $S_1^{-1}(\mathcal{E}_1)$ and $S_1^{-1}(\mathcal{E}_2)$ are the largest volume ellipsoids contained in $S_1^{-1}(K)$. Moreover,

$$\begin{aligned} S_1^{-1}(\mathcal{E}_1) &= S_1^{-1}\big(S_1(\mathrm{B}^n[\mathbf{o},1]) + \mathbf{a}_1\big) \\ &= \mathrm{B}^n[\mathbf{o},1] + S_1^{-1}(\mathbf{a}_1) \end{aligned}$$

and likewise,

$$\begin{aligned} S_1^{-1}(\mathcal{E}_2) &= S_1^{-1}\big(S_1(\mathrm{B}^n[\mathbf{o},1]) + \mathbf{a}_2\big) \\ &= \mathrm{B}^n[\mathbf{o},1] + S_1^{-1}(\mathbf{a}_2). \end{aligned}$$

By *(vi)* of Properties 2.3.1, $\left(S_1^{-1}\right)^{-1} = S_1$ and therefore, S_1^{-1} is invertible. It follows from Lemma 2.4.3 and Theorem 2.4.5 that S_1^{-1} is a bijection. Therefore, if $S^{-1}(\mathbf{a}_1) = S^{-1}(\mathbf{a}_2)$, then $\mathbf{a}_1 = \mathbf{a}_2$, which is a contradiction. Thus, $S_1^{-1}(\mathbf{a}_1) \neq S_1^{-1}(\mathbf{a}_2)$ and hence, $S_1^{-1}(\mathcal{E}_1) \neq S_1^{-1}(\mathcal{E}_2)$.

To simplify the notation, let $K' = S_1^{-1}(K)$, $\mathcal{E}_1' = S_1^{-1}(\mathcal{E}_1)$, $\mathcal{E}_2' = S_1^{-1}(\mathcal{E}_2)$, $\mathbf{a}_1' = S_1^{-1}(\mathbf{a}_1)$ and $\mathbf{a}_2' = S_1^{-1}(\mathbf{a}_2)$. To summarize, $\mathcal{E}_1'$ and $\mathcal{E}_2'$ are distinct closed balls of radius 1 centred at the points $\mathbf{a}_1'$ and $\mathbf{a}_2'$.

Notice that $\operatorname{conv}(\mathcal{E}_1' \cup \mathcal{E}_2') \subseteq K'$. An ellipsoid $\mathcal{E}'$ will be defined so that $\mathcal{E}' \subseteq \operatorname{conv}(\mathcal{E}_1\prime \cup \mathcal{E}_2')$. The line passing through the points $\mathbf{a}_1'$ and $\mathbf{a}_2'$ will be one of the principal axes of $\mathcal{E}'$. Since this line may not be one of the standard axes, use the Principal Axis Theorem to change the coordinate system so that $\mathcal{E}'$ is in standard position. First, notice that the vector $\mathbf{a}_1' - \mathbf{a}_2'$, like any vector in $\mathbb{E}^n$ can be written as a linear combination of the standard basis vectors of $\mathbb{E}^n$: namely, $\mathbf{a}_1' - \mathbf{a}_2' = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2 + \ldots + \mu_n\mathbf{e}_n$ for some $\mu_1, \mu_2, \ldots, \mu_n \in \mathbb{R}$. Since $\mathbf{a}_1' \neq \mathbf{a}_2'$, it follows that $\mathbf{a}_1' - \mathbf{a}_2' \neq 0$. This means that at least one of the terms of the sum $\mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2 + \ldots + \mu_n\mathbf{e}_n$ has a non-zero coefficient; select one of these terms and denote it by $\mu_i\mathbf{e}_i$, where $1 \leqslant \mu_i \leqslant n$. Notice that the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_{i-1}, \mathbf{a}_1' - \mathbf{a}_2', \mathbf{e}_{i+1}, \ldots, \mathbf{e}_n\}$ is linearly independent and spans $\mathbb{E}^n$. Use the Gram-Schmidt orthogonalization method to turn the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_{i-1}, \mathbf{a}_1' - \mathbf{a}_2', \mathbf{e}_{i+1}, \ldots, \mathbf{e}_n\}$ into an orthogonal basis of $\mathbb{E}^n$: Let

$$\begin{aligned}
\mathbf{v}_i &= \mathbf{a}_1' - \mathbf{a}_2'; \\
\mathbf{v}_1 &= \mathbf{e}_1 - \frac{\langle \mathbf{e}_1, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}\mathbf{v}_i; \\
&\vdots \\
\mathbf{v}_{i-1} &= \mathbf{e}_{i-1} - \frac{\langle \mathbf{e}_{i-1}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}\mathbf{v}_i; \\
\mathbf{v}_{i+1} &= \mathbf{e}_{i+1} - \frac{\langle \mathbf{e}_{i+1}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}\mathbf{v}_i; \\
&\vdots \\
\mathbf{v}_n &= \mathbf{e}_n - \frac{\langle \mathbf{e}_n, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}\mathbf{v}_i.
\end{aligned}$$

Normalize each of these vectors and denote this orthonormal basis of $\mathbb{E}^n$ by

$$\{\mathbf{v}_1', \ldots, \mathbf{v}_i', \ldots, \mathbf{v}_n'\}.$$

By *(iv)* of Corollary 3.3.9, that the matrix $P = \begin{bmatrix} \mathbf{v}_1' & \ldots & \mathbf{v}_i' & \ldots & \mathbf{v}_n' \end{bmatrix}$ is an orthogonal matrix. It follows from *(iii)* of Corollary 3.3.9 that $\det(P) = \pm 1$ and therefore, P is invertible. Finally, let

$$A = \begin{bmatrix} 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & \ldots & \frac{1}{\left(\frac{1}{2}\|\mathbf{a}_1' - \mathbf{a}_2'\| + 1\right)^2} & \ldots & 0 \\ \vdots & & \vdots & \ddots & \\ 0 & \ldots & 0 & \ldots & 1 \end{bmatrix}$$

and let

$$\mathcal{E}' = \left\{ \mathbf{x} \in \mathbb{E}^n \;\middle|\; \left\langle \left(\mathbf{x} - \frac{1}{2}\left(\mathbf{a}_1' + \mathbf{a}_2'\right) \right),\ A\left(\mathbf{x} - \frac{1}{2}\left(\mathbf{a}_1' + \mathbf{a}_2'\right) \right) \right\rangle \leqslant 1 \right\},$$

where $\mathbf{x} = P(\mathbf{z})$ for some $\mathbf{z} \in \mathbb{E}^n$.

Thus, $\mathcal{E}' \subseteq \operatorname{conv}(\mathcal{E}_1' \cup \mathcal{E}_2')$ and hence, $\mathcal{E}' \subseteq K'$.

Notice that

$$\operatorname{vol}(\mathcal{E}') = \operatorname{vol}(\mathrm{B}^n[\mathbf{o}, 1]) \left(1 + \frac{1}{2}\|\mathbf{a}_2 - \mathbf{a}_1\| \right).$$

Since $\mathbf{a}_1 \neq \mathbf{a}_2$, it follows that $1 + \frac{1}{2}\|\mathbf{a}_2 - \mathbf{a}_2\| > 1$. This means that

$$\operatorname{vol}(\mathcal{E}') > \operatorname{vol}(\mathrm{B}^n[\mathbf{o}, 1]) = \operatorname{vol}(\mathcal{E}_1') = \operatorname{vol}(\mathcal{E}_2').$$

However, this contradicts that $\mathcal{E}_1'$ and $\mathcal{E}_2'$ are the largest volume ellipsoids in K'. Thus, $\mathbf{a}_1 = \mathbf{a}_2$.

■

Chapter 4

Illuminating Convex Bodies in $\mathbb{E}^3$ with Affine Plane Symmetry

Let K be a convex body. The set of all boundary points of K is denoted by $\mathrm{bd}(K)$. Furthermore, the set of all interior points of K is denoted by $\mathrm{int}(K)$. A direction $\mathbf{d}$ is said to *illuminate* $\mathbf{x} \in \mathrm{bd}(K)$ if

$$r_{\mathbf{d}}^{\mathbf{x}} \cap \mathrm{int}(K) \neq \varnothing$$

where $r_{\mathbf{d}}^{\mathbf{x}} = \{\mathbf{z} \in \mathbb{E}^n \mid \mathbf{z} = \mathbf{x} + \lambda\mathbf{d},\ \lambda \geqslant 0\}$ is the closed ray emanating from $\mathbf{x}$ with direction $\mathbf{d}$. The directions $\mathbf{d_1}, \mathbf{d_2}, \ldots, \mathbf{d_n}$ are said to illuminate K if each boundary point of K is illuminated by at least one of these directions. The minimum number of directions required to illuminate the entire boundary of K is called the illumination number of K.

In 1960, Boltyanski [15] and Hadwiger [27] independently published a conjecture, which is equivalent to the following statement:

Every convex body K in $\mathbb{E}^n$ is illuminated by at most 2^n directions.

This conjecture is, now, known as the Illumination Conjecture. Many partial results towards a complete proof of the Illumination Conjecture have been obtained since 1960. For example, Dekster [21] gave a rough but sound proof of the following theorem:

Theorem 4.1. *If $K \subset \mathbb{E}^3$ is convex body symmetric about a plane, then K can be illuminated by at most eight directions.*

The proof of Theorem 4.1 follows the case structure outlined below.

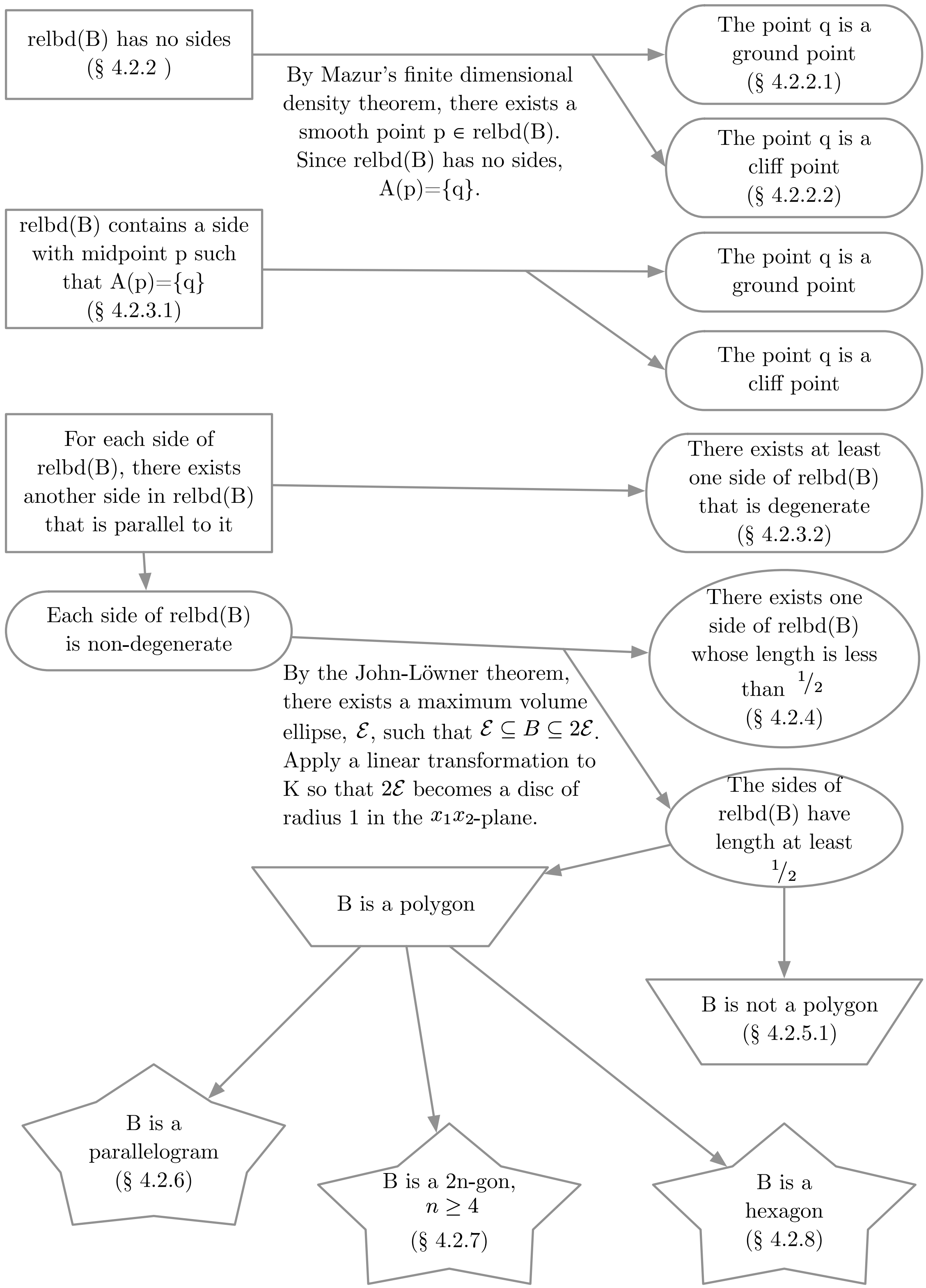

relbd(B) has no sides (§ 4.2.2)
By Mazur's finite dimensional density theorem, there exists a smooth point p ∈ relbd(B). Since relbd(B) has no sides, A(p)={q}.
The point q is a ground point (§ 4.2.2.1)
The point q is a cliff point (§ 4.2.2.2)
relbd(B) contains a side with midpoint p such that A(p)={q} (§ 4.2.3.1)
The point q is a ground point
The point q is a cliff point
For each side of relbd(B), there exists another side in relbd(B) that is parallel to it
There exists at least one side of relbd(B) that is degenerate (§ 4.2.3.2)
Each side of relbd(B) is non-degenerate
There exists one side of relbd(B) whose length is less than $^1/_2$ (§ 4.2.4)
By the John-Löwner theorem, there exists a maximum volume ellipse, $\mathcal{E}$, such that $\mathcal{E} \subseteq B \subseteq 2\mathcal{E}$. Apply a linear transformation to K so that $2\mathcal{E}$ becomes a disc of radius 1 in the x_1x_2-plane.
The sides of relbd(B) have length at least $^1/_2$
B is a polygon
B is not a polygon (§ 4.2.5.1)
B is a parallelogram (§ 4.2.6)
B is a 2n-gon, $n \geq 4$ (§ 4.2.7)
B is a hexagon (§ 4.2.8)

Figure 4.1

4.1 Preliminaries

A set S in $\mathbb{E}^3$ is *affine plane symmetric* about some plane H with respect to the line L if the following two conditions are met:

(**a**) L meets H at exactly one point; and

(**b**) for any $\mathbf{s} \in S$, there exists a vector $\mathbf{t} \in \mathbb{E}^3$ and a point $\mathbf{s}' \in S$ such that $\mathbf{s}, \mathbf{s}' \in L + \mathbf{t}$ and $\frac{1}{2}(\mathbf{s} + \mathbf{s}') \in H \cap S$.

Note that the line L and the plane H need not be orthogonal.

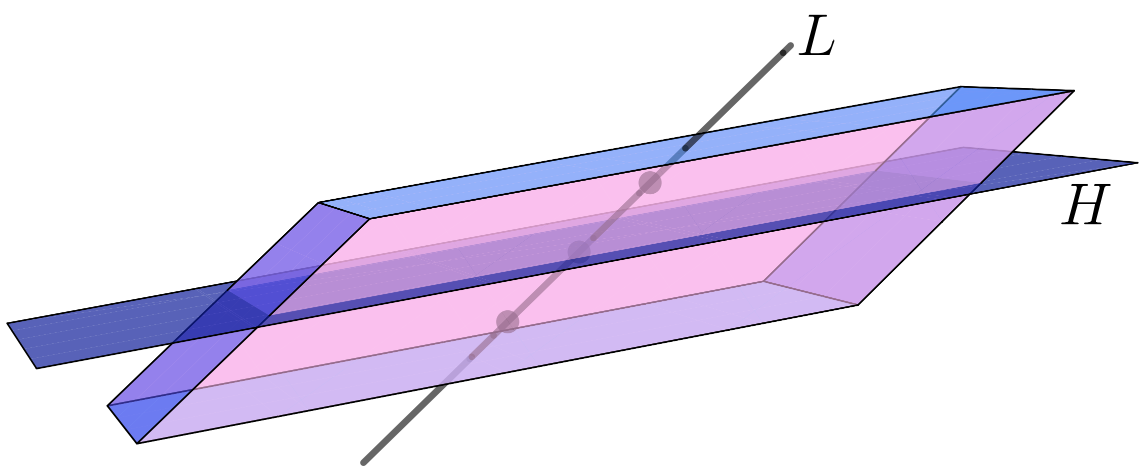


Figure 4.2: The parallelepiped is affine plane symmetric about the plane H with respect to the line L but H and L are not orthogonal.

Let x_1, x_2 and x_3 denote the axes in $\mathbb{E}^3$ and let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ denote the standard basis vectors of $\mathbb{E}^3$.

Due to the fact that the illumination number of a convex body is invariant under rotation and translation, all affine plane symmetric convex bodies K in $\mathbb{E}^3$ are hereinafter assumed, without loss of generality, to be affine plane symmetric about the x_1x_2-plane with respect to some line L. Projections onto the x_1x_2-plane are simpler when L is orthogonal to the x_1x_2-plane. For convenience, apply the following transformation, T, to K; it will ensure that the line L is orthogonal to the x_1x_2-plane. Let

$$T(\mathbf{e}_1) = \mathbf{e}_1$$

$$T(\mathbf{e}_2) = \mathbf{e}_2$$

$$T(\mathbf{u}) = \mathbf{e}_3,$$

where $\mathbf{u}$ is the unit vector with non-negative coordinates such that the line $\{\lambda\mathbf{u} \mid \lambda \in \mathbb{R}\}$ is parallel to L. Notice that for any $\mathbf{z} \in \mathbb{E}^3$,

$$T(\mathbf{z}) = \begin{bmatrix} 1 & 0 & -\dfrac{\langle \mathbf{u}, \mathbf{e}_1 \rangle}{\langle \mathbf{u}, \mathbf{e}_3 \rangle} \\ 0 & 1 & -\dfrac{\langle \mathbf{u}, \mathbf{e}_2 \rangle}{\langle \mathbf{u}, \mathbf{e}_3 \rangle} \\ 0 & 0 & \dfrac{1}{\langle \mathbf{u}, \mathbf{e}_3 \rangle} \end{bmatrix} \mathbf{z}.$$

It readily follows that the transformation T preserves vector addition and scalar multiplication. Thus, T is a linear transformation. The illumination number of a convex body is also invariant under any linear transformation. Therefore, affine plane symmetric convex bodies K in $\mathbb{E}^3$ are hereinafter assumed to be affine plane symmetric about the x_1x_2-plane with respect to a line L, which is orthogonal to the x_1x_2-plane.

Let $\mathbf{z} = (z_1, z_2, z_3)$ be some arbitrary vector of $\mathbb{E}^3$. The map $\mathrm{Pr} : \mathbb{E}^3 \to \mathbb{E}^2 \times \{0\}$, which is called the *orthogonal projection onto the x_1x_2-plane*, sends $\mathbf{z}$ to the vector $\mathrm{Pr}(\mathbf{z}) = (z_1, z_2, 0)$ in $\mathbb{E}^2 \times \{0\}$. The orthogonal projection of any subset S of $\mathbb{E}^3$ onto the x_1x_2-plane is defined to be

$$\mathrm{Pr}(S) = \{\langle \mathbf{s}, \mathbf{e_1} \rangle \, \mathbf{e_1} + \langle \mathbf{s}, \mathbf{e_2} \rangle \, \mathbf{e_2} \mid \mathbf{s} \in S\}.$$

If $S \subseteq \mathbb{E}^n$ is a convex and affine plane symmetric set, then its orthogonal projection onto the x_1x_2-plane is simply its intersection with the x_1x_2-plane, that is $S \cap (\mathbb{E}^2 \times \{0\})$. In particular, the projection of K onto the x_1x_2-plane is equivalent to $K \cap (\mathbb{E}^2 \times \{0\})$ and will be referred to as the *base set*, B, of K. The base set, B, has the following properties:

Properties 4.1.1.

(i) *B is a convex body in the x_1x_2-plane;*

(ii) relbd$(B) \subseteq$ bd(K)*;*

(iii) relint$(B) =$ Pr(int(K)).

Proof. **(i)** Recall that all affine sets are convex and closed. Thus, the x_1x_2-plane, $\mathbb{E}^2 \times \{0\}$, is closed and convex. Moreover, K is closed and convex. Thus, B is closed and convex since it is the intersection of two closed, convex sets. The base set B is a subset of K, which is bounded. Thus, B is bounded. It follows that B is convex and compact. Since K is a convex body, it follows that int$(K) \neq \varnothing$. This implies that $K \neq \varnothing$. Moreover, K is affine plane symmetric about the x_1x_2-plane. It follows from the definition of affine plane symmetry, given above, that a subset of $K \cap (\mathbb{E}^2 \times \{0\}) = B$ is non-empty. Thus, $B \neq \varnothing$. Now, it follows that relint$(B) \neq \varnothing$ (see Theorem 2.3.1 in [60]). Furthermore, note that aff$(B) = \mathbb{E}^2 \times \{0\}$. In other words, B lies entirely in $\mathbb{E}^2 \times \{0\}$. Hence, B is a convex body in the x_1x_2-plane.

(ii) Let $\mathbf{x} \in$ relbd(B) be arbitrarily chosen. It follows, by definition, that

$$\text{relbd}(B) = \text{cl}(B) \backslash \text{relint}(B)$$

where the sets relbd(B) and relint(B) are disjoint. This means $\mathbf{x} \in$ cl(B) and $\mathbf{x} \notin$ relint(B). In **(i)**, it was established that B is closed; therefore, $\mathbf{x} \in \text{cl}(B) = B \subseteq K$. It follows from K being closed and Theorem 2.5.7 that

$$K = \text{int}(K) \cup \text{bd}(K)$$

where int$(K) \cap$ bd$(K) = \varnothing$. This means that either $\mathbf{x} \in$ int(K) or $\mathbf{x} \in$ bd(K). Suppose for a contradiction that $\mathbf{x} \in$ int(K). Then, there exists a real number $r > 0$ such that

$$\text{B}(\mathbf{x}, r) \subseteq K.$$

It follows that

$$\mathrm{B}(\mathbf{x}, r) \cap \big(\mathbb{E}^2 \times \{\mathbf{o}\}\big) \subseteq K \cap \big(\mathbb{E}^2 \times \{\mathbf{o}\}\big) = B.$$

Notice that $\mathrm{aff}(B) = \mathbb{E}^2 \times \{\mathbf{o}\}$. It follows, by definition, that $\mathbf{x} \in \mathrm{relint}(B)$. However, this is a contradiction. Therefore, $\mathbf{x} \in \mathrm{bd}(K)$, which implies that $\mathrm{relbd}(B) \subseteq \mathrm{bd}(K)$.

(iii) First, it will be shown that $\mathrm{Pr}(\mathrm{int}(K)) \subseteq \mathrm{relint}(B)$. Let $\mathbf{x} \in \mathrm{Pr}(\mathrm{int}(K))$ be arbitrarily chosen. Then, there exists some $\mathbf{k} \in \mathrm{int}(K) \subseteq K$ such that

$$\mathbf{x} = \langle \mathbf{k}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{k}, \mathbf{e}_2 \rangle \mathbf{e}_2.$$

Since K is affine plane symmetric about $\mathbb{E}^2 \times \{\mathbf{o}\}$, it follows that there exists $\mathbf{k}' \in K$ such that $\mathbf{k}' = \mathbf{k} + \mu \mathbf{e}_3$ for some $\mu \in \mathbb{R}$ and $\frac{1}{2}(\mathbf{k} + \mathbf{k}') \in B = K \cap \big(\mathbb{E}^2 \times \{\mathbf{o}\}\big)$. Notice that

$$\begin{aligned}
\frac{1}{2}(\mathbf{k} + \mathbf{k}') &= \left\langle \frac{1}{2}(\mathbf{k} + \mathbf{k}'), \mathbf{e}_1 \right\rangle \mathbf{e}_1 + \left\langle \frac{1}{2}(\mathbf{k} + \mathbf{k}'), \mathbf{e}_2 \right\rangle \mathbf{e}_2 \\
&= \left\langle \mathbf{k} + \frac{\mu}{2}\mathbf{e}_3, \mathbf{e}_1 \right\rangle \mathbf{e}_1 + \left\langle \mathbf{k} + \frac{\mu}{2}\mathbf{e}_3, \mathbf{e}_2 \right\rangle \mathbf{e}_2 \\
&= \langle \mathbf{k}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{k}, \mathbf{e}_2 \rangle \mathbf{e}_2 + \left\langle \frac{\mu}{2}\mathbf{e}_3, \mathbf{e}_1 \right\rangle \mathbf{e}_1 + \left\langle \frac{\mu}{2}\mathbf{e}_3, \mathbf{e}_2 \right\rangle \mathbf{e}_2 \\
&= \mathbf{x}.
\end{aligned}$$

Therefore, $\mathbf{x} \in B$. In **(i)**, it was shown that B is closed. This means that $B = \mathrm{relint}(B) \cup \mathrm{relbd}(B)$. Note that $\mathrm{relint}(B) \cap \mathrm{relbd}(B) = \varnothing$ since the relative interior and relative boundary of any set is disjoint. In particular, this means that either $\mathbf{x} \in \mathrm{relint}(B)$ or $\mathbf{x} \in \mathrm{relbd}(B)$. Suppose for a contradiction that $\mathbf{x} \in \mathrm{relbd}(B)$. In **(ii)**, it was shown that $\mathrm{relbd}(B) \subseteq \mathrm{bd}(K)$. Thus, $\mathbf{x} \in \mathrm{bd}(K)$. By Corollary 2.10.11,

$$[\mathbf{k}, \mathbf{k}') \subseteq \mathrm{int}(K).$$

However, $\mathbf{x} = \frac{1}{2}(\mathbf{k} + \mathbf{k}') \in [\mathbf{k}, \mathbf{k}') \subseteq \mathrm{int}(K)$. This is a contradiction. Hence, $\mathbf{x} \in \mathrm{relint}(B)$, implying that $\mathrm{Pr}(\mathrm{int}(K)) \subseteq \mathrm{relint}(B)$.

Finally, it will be shown that $\mathrm{relint}(B) \subseteq \Pr(\mathrm{int}(K))$. Let $\mathbf{x} \in \mathrm{relint}(B)$ be arbitrarily chosen. Then, there exists some real number $r > 0$ such that

$$\mathrm{B}(\mathbf{x}, r) \cap \left(\mathbb{E}^2 \times \{\mathbf{o}\}\right) \subseteq B.$$

Recall that K is a convex body, which implies $\mathrm{int}(K) \neq \varnothing$. This means that there exists $\mathbf{k} \in \mathrm{int}(K) \subseteq K$. Also, recall that K is affine plane symmetric about $\mathbb{E}^2 \times \{\mathbf{o}\}$, meaning that there exists $\mathbf{k}' \in K$ such that $\mathbf{k}' = \mathbf{k} + \mu\mathbf{e}_3$ for some $\mu \in \mathbb{R}$ and $\frac{1}{2}(\mathbf{k} + \mathbf{k}') \in B$. Notice that

$$\begin{aligned}
\frac{1}{2}(\mathbf{k} + \mathbf{k}') &= \left\langle \mathbf{k} + \frac{\mu}{2}\mathbf{e}_3, \mathbf{e}_1 \right\rangle \mathbf{e}_1 + \left\langle \mathbf{k} + \frac{\mu}{2}\mathbf{e}_3, \mathbf{e}_2 \right\rangle \mathbf{e}_2 \\
&= \langle \mathbf{k}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{k}, \mathbf{e}_2 \rangle \mathbf{e}_2 = \Pr(\mathbf{k}).
\end{aligned}$$

By Corollary 2.10.11, $[\mathbf{k}, \mathbf{k}') \subseteq \mathrm{int}(K)$. Thus, $\Pr(\mathbf{k}) \in \mathrm{int}(K)$. This means that there exists a real number $r' > 0$ such that

$$\mathrm{B}(\Pr(\mathbf{k}), r') \subseteq K.$$

Let $\mathbf{z} \in \mathrm{B}\left(\Pr(\mathbf{k}) + \frac{r'}{2}\mathbf{e}_3, \frac{r'}{2}\right)$ be arbitrarily chosen. Then, there exists some real number $0 \leqslant \mu' \leqslant 1$ and unit vector $\mathbf{u}$ such that

$$\begin{aligned}
\mathbf{z} &= \left(\Pr(\mathbf{k}) + \frac{r'}{2}\mathbf{e}_3\right) + \mu' \cdot \frac{r'}{2}\mathbf{u} \\
&= \mathbf{k} + r'\left(\frac{\|\mathbf{e}_3 + \mu'\mathbf{u}\|}{2}\right)\left(\frac{\mathbf{e}_3 + \mu'\mathbf{u}}{\|\mathbf{e}_3 + \mu'\mathbf{u}\|}\right).
\end{aligned}$$

The triangle inequality, properties of the Euclidean norm, the bounds of μ', and the fact that $\mathbf{e}_3$ and $\mathbf{u}$ are unit vectors together imply that

$$0 \leqslant \|\mathbf{e}_3 + \mu'\mathbf{u}\| \leqslant \|\mathbf{e}_3\| + \mu'\|\mathbf{u}\| = 1 + \mu' < 2.$$

It immediately follows that $\mathbf{z} \in \mathrm{B}(\mathbf{k}, r')$. Therefore,

$$\mathrm{B}\left(\Pr(\mathbf{k}) + \frac{r'}{2}\mathbf{e}_3, \frac{r'}{2}\right) \subseteq \mathrm{B}(\mathbf{k}, r') \subseteq K.$$

This implies that $\Pr(\mathbf{k}) + \frac{r'}{2}\mathbf{e}_3 \in \operatorname{int}(K)$.

If $\Pr(\mathbf{k}) = \mathbf{x}$, then

$$\begin{aligned}
&\left\langle \Pr(\mathbf{k}) + \frac{r'}{2}\mathbf{e}_3, \mathbf{e}_1 \right\rangle \mathbf{e}_1 + \left\langle \Pr(\mathbf{k}) + \frac{r'}{2}\mathbf{e}_3, \mathbf{e}_2 \right\rangle \mathbf{e}_2 \\
&= \langle \Pr(\mathbf{k}), \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \Pr(\mathbf{k}), \mathbf{e}_2 \rangle \mathbf{e}_2 \\
&= \Pr(\mathbf{k}) = \mathbf{x}.
\end{aligned}$$

This means that $\mathbf{x} \in \Pr(\operatorname{int}(K))$, which implies that $\operatorname{relbd}(B) \subseteq \Pr(\operatorname{int}(K))$. For the remainder of the argument, suppose that $\Pr(\mathbf{k}) \neq \mathbf{x}$. The affine combination of the vectors $\mathbf{x}, \Pr(\mathbf{k}) \in \mathbb{E}^2 \times \{\mathbf{o}\}$,

$$\left(1 + \frac{r}{2} \cdot \frac{1}{\|\mathbf{x} - \Pr(\mathbf{k})\|}\right)\mathbf{x} + \left(-\frac{r}{2} \cdot \frac{1}{\|\mathbf{x} - \Pr(\mathbf{k})\|}\right)\Pr(\mathbf{k}),$$

can be re-written as

$$\mathbf{x} + \frac{r}{2}\left(\frac{\mathbf{x} - \Pr(\mathbf{k})}{\|\mathbf{x} - \Pr(\mathbf{k})\|}\right) \in \mathrm{B}(\mathbf{x}, r) \cap \left(\mathbb{E}^2 \times \{\mathbf{o}\}\right) \subseteq B \subseteq K.$$

Corollary 2.10.11 implies that

$$\left[\Pr(\mathbf{k}) + \frac{r'}{2}\mathbf{e}_3, \mathbf{x} + \frac{r}{2}\left(\frac{\mathbf{x} - \Pr(\mathbf{k})}{\|\mathbf{x} - \Pr(\mathbf{k})\|}\right)\right) \subseteq \operatorname{int}(K).$$

Recall that the Euclidean norm is always non-negative. However,

$$\|\mathbf{x} - \Pr(\mathbf{k})\| > 0$$

since $\Pr(\mathbf{k}) \neq \mathbf{x}$. This implies that $2\|\mathbf{x} - \Pr(\mathbf{k})\| > 0$. Recall that $r > 0$, so $r + 2\|\mathbf{x} - \Pr(\mathbf{k})\| > 0$. Notice that

$$0 < 2\|\mathbf{x} - \Pr(\mathbf{k})\| = (r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|)\left(\frac{2\|\mathbf{x} - \Pr(\mathbf{k})\|}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|}\right).$$

Recall that a positive number can either be written as the product of two positive numbers or two negative numbers. Therefore,

$$\frac{2\|\mathbf{x} - \Pr(\mathbf{k})\|}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} > 0.$$

Also, notice that

$$
\begin{aligned}
0 < r &= (r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|) \left(\frac{r}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} \right) \\
&= (r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|) \left(1 - \frac{2\|\mathbf{x} - \Pr(\mathbf{k})\|}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} \right).
\end{aligned}
$$

It follows that

$$
1 - \frac{2\|\mathbf{x} - \Pr(\mathbf{k})\|}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} > 0,
$$

which implies that

$$
\frac{2\|\mathbf{x} - \Pr(\mathbf{k})\|}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} < 1.
$$

Observe that the point

$$
\frac{2\|\mathbf{x} - \Pr(\mathbf{k})\|}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} \left(\mathbf{x} + \frac{r}{2} \left(\frac{\mathbf{x} - \Pr(\mathbf{k})}{\|\mathbf{x} - \Pr(\mathbf{k})\|} \right) \right) + \frac{r}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} \left(\Pr(\mathbf{k}) + \frac{r'}{2} \mathbf{e}_3 \right)
$$

in the half-open segment

$$
\left[\Pr(\mathbf{k}) + \frac{r'}{2} \mathbf{e}_3, \mathbf{x} + \frac{r}{2} \left(\frac{\mathbf{x} - \Pr(\mathbf{k})}{\|\mathbf{x} - \Pr(\mathbf{k})\|} \right) \right)
$$

is equivalent to

$$
\begin{aligned}
&\left(\frac{2\|\mathbf{x} - \Pr(\mathbf{k})\|}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} + \frac{r}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} \right) \mathbf{x} + \\
&\quad \left(- \frac{r}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} + \frac{r}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} \right) \Pr(\mathbf{k}) + \frac{rr'}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} \mathbf{e}_3 \\
&= \mathbf{x} + \frac{rr'}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} \mathbf{e}_3.
\end{aligned}
$$

Therefore,

$$
\mathbf{x} + \frac{rr'}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} \mathbf{e}_3 \in \operatorname{int}(K)
$$

and

$$
\begin{aligned}
&\left\langle \mathbf{x} + \frac{rr'}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} \mathbf{e}_3, \mathbf{e}_1 \right\rangle \mathbf{e}_1 + \left\langle \mathbf{x} + \frac{rr'}{r + 2\|\mathbf{x} - \Pr(\mathbf{k})\|} \mathbf{e}_3, \mathbf{e}_2 \right\rangle \mathbf{e}_2 \\
&= \langle \mathbf{x}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{x}, \mathbf{e}_2 \rangle \mathbf{e}_2 = \mathbf{x}.
\end{aligned}
$$

Thus, $\mathbf{x} \in \Pr(\operatorname{int}(K))$. Hence, $\operatorname{relbd}(B) \subseteq \operatorname{int}(K)$.

$\blacksquare$

An immediate consequence of **(iii)** from Properties 4.1.1 is that $\mathrm{relint}(B) \subseteq \mathrm{int}(K)$.

Let $\mathbf{z}$ be any arbitrary vector in the x_1x_2-plane. The *pre-image* of the projection map onto the x_1x_2-plane $\mathrm{Pr}^{-1} : \mathbb{E}^2 \times \{0\} \to \mathbb{E}^3$ sends the vector $\mathbf{z}$ to the line $\{\ \mathbf{z} + \lambda\mathbf{e}_3 \mid \lambda \in \mathbb{R}\}$. The pre-image of any set S lying completely in the x_1x_2-plane is defined to be

$$\begin{aligned} Pr^{-1}(S) &= S + \{\lambda\mathbf{e}_3 \mid \lambda \in \mathbb{R}\} \\ &= \{\mathbf{z} \in \mathbb{E}^3 \mid \langle\mathbf{z}, \mathbf{e_1}\rangle\mathbf{e_1} + \langle\mathbf{z}, \mathbf{e_2}\rangle\mathbf{e_2} = \mathrm{Pr}(\mathbf{z}) \in S\}. \end{aligned}$$

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{E}^3$. The closed line segment between these two vectors is denoted by $[\mathbf{a}, \mathbf{b}] = \{\mathbf{z} \in \mathbb{E}^3 \mid \mathbf{z} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}),\ 0 \leqslant \lambda \leqslant 1\}$. The open line segment is denoted by $(\mathbf{a}, \mathbf{b}) = \{\mathbf{z} \in \mathbb{E}^3 \mid \mathbf{z} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}),\ 0 < \lambda < 1\}$. Finally, the half-open, or half-closed, line segments between these two vectors is denoted by either $[\mathbf{a}, \mathbf{b}) = \{\mathbf{z} \in \mathbb{E}^3 \mid \mathbf{z} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}),\ 0 \leqslant \lambda < 1\}$ or $(\mathbf{a}, \mathbf{b}]$. The length of all the above line segments is defined as $\|\mathbf{a} - \mathbf{b}\| = \|\mathbf{b} - \mathbf{a}\|$. If the vectors $\mathbf{a}$ and $\mathbf{b}$ lie on some planar curve C, then $[\mathbf{a}, \mathbf{b}]_C, (\mathbf{a}, \mathbf{b})_C, [\mathbf{a}, \mathbf{b})_C, (\mathbf{a}, \mathbf{b}]_C$ respectively denote closed, open and half-open, or half-closed, arcs of the curve C with positive orientation. Specifically, the reader will encounter the notation $[\mathbf{a}, \mathbf{b}]_B, (\mathbf{a}, \mathbf{b})_B, [\mathbf{a}, \mathbf{b})_B, (\mathbf{a}, \mathbf{b}]_B$, by which the closed, open, half-open and half-closed arcs of the relative boundary of B, $\mathrm{relbd}(B)$, with positive orientation are respectively meant.

The closed halfspace $H_{+} = \{\mathbf{z} \in \mathbb{E}^3 \mid \langle\mathbf{z}, \mathbf{e_3}\rangle \geqslant 0\}$ represents the region on and above the x_1x_2-plane. Likewise, the closed halfspace $H_{-} = \{\mathbf{z} \in \mathbb{E}^3 \mid \langle\mathbf{z}, \mathbf{e_3}\rangle \leqslant 0\}$ represents the region on and below the x_1x_2-plane.

With the foregoing definitions in place, it is now possible give the following definitions, which will be used extensively in the proof of Theorem 4.1. For any subset X of the base set B of K,

$$X_{+} = \mathrm{Pr}^{-1}(X) \cap \mathrm{bd}(K) \cap H_{+} \quad \text{and} \quad X_{-} = \mathrm{Pr}^{-1}(X) \cap \mathrm{bd}(K) \cap H_{-}.$$

The *wall* of K, which we denote by W, is $\mathrm{Pr}^{-1}(\mathrm{relbd}(B)) \cap \mathrm{bd}(K)$.

Proposition 4.1.2. *The three sets* W*,* $\operatorname{relint}(B)_+$ *and* $\operatorname{relint}(B)_-$ *are pairwise disjoint and* $\operatorname{bd}(K) = W \cup \operatorname{relint}(B)_+ \cup \operatorname{relint}(B)_-$.

Proof. First, it will be shown that W, $\operatorname{relint}(B)_+$ and $\operatorname{relint}(B)_-$ are pairwise disjoint.

Suppose for a contradiction that the sets $\operatorname{relint}(B)_+$ and $\operatorname{relint}(B)_-$ are not disjoint. This means that there exists an element $\mathbf{x} \in \mathbb{E}^3$ such that $\mathbf{x} \in \operatorname{relint}(B)_+ \cap \operatorname{relint}(B)_-$. It follows that $\mathbf{x} \in H_+ \cap H_- \cap \operatorname{bd}(K)$ and $\operatorname{Pr}(\mathbf{x}) \in \operatorname{relint}(B)$. Notice that

$$\begin{aligned} H_+ \cap H_- &= \left\{\mathbf{z} \in \mathbb{E}^3 \mid \langle \mathbf{z}, \mathbf{e}_3 \rangle \geqslant 0\right\} \cap \left\{\mathbf{z} \in \mathbb{E}^3 \mid \langle \mathbf{z}, \mathbf{e}_3 \rangle \leqslant 0\right\} \\ &= \left\{\mathbf{z} \in \mathbb{E}^3 \mid \langle \mathbf{z}, \mathbf{e}_3 \rangle = 0\right\} = \mathbb{E}^2 \times \{\mathbf{o}\}. \end{aligned}$$

Since $\mathbf{x} \in \mathbb{E}^2 \times \{\mathbf{o}\}$, it follows that $\mathbf{x} = \operatorname{Pr}(\mathbf{x})$. Therefore, $\mathbf{x} \in \operatorname{relint}(B) \subseteq \operatorname{int}(K)$ and $\mathbf{x} \in \operatorname{bd}(K)$. However, it follows Theorem 2.5.7 that $\operatorname{int}(K) \cap \operatorname{bd}(K) = \varnothing$, which is a contradiction. Hence, $\operatorname{relint}(B)_+ \cap \operatorname{relint}(B)_- = \varnothing$.

Suppose for a contradiction that the sets W and $\operatorname{relint}(B)_+$ are not disjoint. This means that there exists an element $\mathbf{x} \in \mathbb{E}^3$ such that $\mathbf{x} \in W \cap \operatorname{relint}(B)_+$. It follows that $\operatorname{Pr}(\mathbf{x}) \in \operatorname{relbd}(B)$ and $\operatorname{Pr}(\mathbf{x}) \in \operatorname{relint}(B)$. However, by definition, the relative boundary and relative interior of any set are disjoint, which is a contradiction. Hence, $W \cap \operatorname{relint}(B)_+ = \varnothing$.

A very similar argument can be used to show that $W \cap \operatorname{relint}(B)_- = \varnothing$.

To show that $\operatorname{bd}(K) = W \cup \operatorname{relint}(B)_+ \cup \operatorname{relint}(B)_-$, first show that $W \cup \operatorname{relint}(B)_+ \cup \operatorname{relint}(B)_- \subseteq \operatorname{bd}(K)$ and then show that $\operatorname{bd}(K) \subseteq W \cup \operatorname{relint}(B)_+ \cup \operatorname{relint}(B)_-$.

Suppose that $\mathbf{x} \in W \cup \operatorname{relint}(B)_+ \cup \operatorname{relint}(B)_-$. Notice that $W = \operatorname{Pr}^{-1}(\operatorname{relbd}(B)) \cap \operatorname{bd}(K) \subseteq \operatorname{bd}(K)$ and $\operatorname{relint}(B)_+ = \operatorname{Pr}^{-1}(\operatorname{relint}(B)) \cap \operatorname{bd}(K) \cap H_+ \subseteq \operatorname{bd}(K)$. Likewise, $\operatorname{relint}(B)_- \subseteq \operatorname{bd}(K)$. It follows that $\mathbf{x} \in \operatorname{bd}(K)$. Thus, $W \cup \operatorname{relint}(B)_+ \cup \operatorname{relint}(B)_- \subseteq \operatorname{bd}(K)$.

Suppose that $\mathbf{x} \in \operatorname{bd}(K) \subseteq K$. Since K is affine plane symmetric, $\operatorname{Pr}(x) \in \operatorname{Pr}(K) = B$. Recall from Properties 4.1.1 **i** that B is closed. It follows that $B = \operatorname{relbd}(B) \cup \operatorname{relint}(B)$, where the sets $\operatorname{relbd}(B)$ and $\operatorname{relint}(B)$ are disjoint. Then, either $\operatorname{Pr}(\mathbf{x}) \in \operatorname{relbd}(B)$ or $\operatorname{Pr}(\mathbf{x}) \in \operatorname{relint}(B)$.

<u>Case 1</u>: Suppose $\operatorname{Pr}(\mathbf{x}) \in \operatorname{relbd}(B)$.

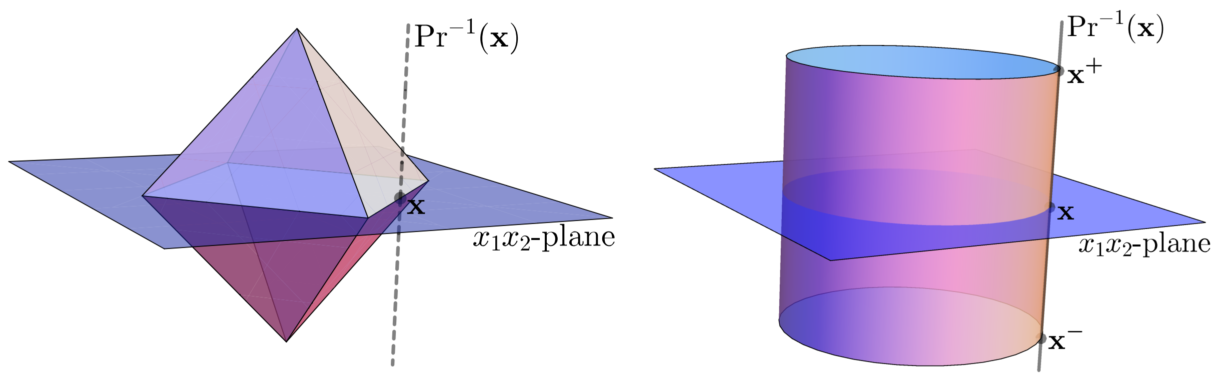
It follows, by definition, that $\mathbf{x} \in \mathrm{Pr}^{-1}(\mathrm{relbd}(B))$. Also, recall that $\mathbf{x} \in \mathrm{bd}(K)$, by supposition. This means $\mathbf{x} \in W$. Therefore, $\mathbf{x} \in W \cup \mathrm{relint}(B)_+ \cup \mathrm{relint}(B)_-$.

Case 2: Suppose $\mathrm{Pr}(\mathbf{x}) \in \mathrm{relint}(B)$.

It follows, by definition, that $\mathbf{x} \in \mathrm{Pr}^{-1}(\mathrm{relint}(B))$. Again, $\mathbf{x} \in \mathrm{bd}(K)$, by supposition. Note that $\mathbf{x} \in H_+$ or $\mathbf{x} \in H_-$. It follows that if $\mathbf{x} \in H_+$, then $\mathbf{x} \in \mathrm{relint}(B)_+$ and if $\mathbf{x} \in H_-$, then $\mathbf{x} \in \mathrm{relint}(B)_-$. Thus, $\mathbf{x} \in W \cup \mathrm{relint}(B)_+ \cup \mathrm{relint}(B)_-$.

Together Case 1 and Case 2 imply that $\mathrm{bd}(K) \subseteq W \cup \mathrm{relint}(B)_+ \cup \mathrm{relint}(B)_-$. ■

For any subset Y of $\mathrm{relbd}(B)$, the *wall through* Y, which we denote by W_Y, is $\mathrm{Pr}^{-1}(Y) \cap \mathrm{bd}(K)$. A vector $\mathbf{x} \in \mathrm{relbd}(B)$ is called a *ground point* if $\mathrm{Pr}^{-1}(\mathbf{x}) \cap \mathrm{bd}(K) = \{\mathbf{x}\}$. If a vector $\mathbf{x} \in \mathrm{relbd}(B)$ is not a ground point, it is said to be a *cliff point*. In other words, $\mathbf{x} \in \mathrm{relbd}(B)$ is a cliff point if $\mathrm{Pr}^{-1}(\mathbf{x}) \cap \mathrm{bd}(K)$ is some non-degenerate line segment. Often, this non-degenerate line segment will be referred to as the *cliff* at $\mathbf{x}$ and will be denoted by $[\mathbf{x}^-, \mathbf{x}^+]$.



The point $\mathbf{x} \in \mathrm{relbd}(B)$ is a ground point. Each side of B is degenerate.

The point $\mathbf{x} \in \mathrm{relbd}(B)$ is a cliff point. All points on the relative boundary of B are cliff points

Figure 4.3

Let S be some convex set and let H be some supporting hyperplane of S. The set $H \cap S$ is called an *exposed face* of S. Any exposed 1-dimensional face of B in $\mathbb{E}^2 \times \{0\}$ is called a *side* of B. Informally, a side is a non-degenerate closed segment in $\mathrm{relbd}(B)$, which is not a part of another such segment.

Proposition 4.1.3. *Let $Y \subseteq \mathrm{relbd}(B)$ be a closed line segment. Then W_Y is a compact, convex set which is symmetric about Y.*

Proof. First, it will be shown that W_Y is symmetric about Y.

Let $\mathbf{x} \in W_Y$ be arbitrarily chosen. Recall that $W_Y = \mathrm{Pr}^{-1}(Y) \cap \mathrm{bd}(K)$. It follows that $\mathbf{x} \in \mathrm{Pr}^{-1}(Y)$. By definition, $\mathrm{Pr}(\mathbf{x}) \in Y$. Also, notice that $W_Y \subseteq K$. This means that $\mathbf{x} \in K$. Since K is affine plane symmetric about the x_1x_2-plane, there exists $\mathbf{x}' \in K$ such that $\mathbf{x}' = \mathbf{x} + \mu\mathbf{e}_3$, for some $\mu \in \mathbb{R}$ and

$$\frac{1}{2}(\mathbf{x} + \mathbf{x}') \in B. \tag{4.1}$$

Substitute $\mathbf{x}' = \mathbf{x} + \mu\mathbf{e}_3$ into (4.1) to get that

$$\frac{1}{2}(\mathbf{x} + \mathbf{x}') = \mathbf{x} + \frac{\mu}{2}\mathbf{e}_3. \tag{4.2}$$

It follows from the fact that $\frac{1}{2}(\mathbf{x} + \mathbf{x}') \in \mathbb{E}^2 \times \{\mathbf{o}\}$ that

$$\frac{1}{2}(\mathbf{x} + \mathbf{x}') = \left\langle \frac{1}{2}(\mathbf{x} + \mathbf{x}'), \mathbf{e}_1 \right\rangle \mathbf{e}_1 + \left\langle \frac{1}{2}(\mathbf{x} + \mathbf{x}'), \mathbf{e}_2 \right\rangle \mathbf{e}_2$$

and by (4.2)

$$\begin{aligned} &= \left\langle \mathbf{x} + \frac{\mu}{2}\mathbf{e}_3, \mathbf{e}_1 \right\rangle \mathbf{e}_1 + \left\langle \mathbf{x} + \frac{\mu}{2}\mathbf{e}_3, \mathbf{e}_2 \right\rangle \mathbf{e}_2 \\ &= \langle \mathbf{x}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{x}, \mathbf{e}_2 \rangle \mathbf{e}_2 + \frac{\mu}{2}\langle \mathbf{e}_3, \mathbf{e}_1 \rangle \mathbf{e}_1 + \frac{\mu}{2}\langle \mathbf{e}_3, \mathbf{e}_2 \rangle \mathbf{e}_2 \\ &= \langle \mathbf{x}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{x}, \mathbf{e}_2 \rangle \mathbf{e}_2 = \mathrm{Pr}(\mathbf{x}). \end{aligned}$$

Therefore, $\frac{1}{2}(\mathbf{x} + \mathbf{x}') \in Y$.

Since K is closed, it follows from Theorem 2.5.7 that $K = \mathrm{bd}(K) \cup \mathrm{int}(K)$ where the sets $\mathrm{bd}(K)$ and $\mathrm{int}(K)$ are disjoint. This means that either $\mathbf{x}' \in \mathrm{bd}(K)$ or $\mathbf{x}' \in \mathrm{int}(K)$. Suppose for a contradiction that $\mathbf{x}' \in \mathrm{int}(K)$. Then, it follows from Theorem 2.10.10 that $[\mathbf{x}', \mathbf{x}) \subseteq \mathrm{int}(K)$. However, $\frac{1}{2}(\mathbf{x} + \mathbf{x}') \in [\mathbf{x}', \mathbf{x})$ and it was shown above that $\frac{1}{2}(\mathbf{x} + \mathbf{x}') \in Y \subseteq \mathrm{relbd}(B) \subseteq \mathrm{bd}(K)$, which is a contradiction. Thus, $\mathbf{x}' \in \mathrm{bd}(K)$. Note that for some $\mu \in \mathbb{R}$

$$\mathbf{x}' = \mathbf{x} + \mu\mathbf{e}_3$$

$$= \mathbf{x} + \frac{\mu}{2}\mathbf{e}_3 + \frac{\mu}{2}\mathbf{e}_3$$
$$= \mathrm{Pr}(\mathbf{x}) + \frac{\mu}{2}\mathbf{e}_3 \in Y + \{\lambda \mathbf{e}_3 \mid \lambda \in \mathbb{R}\} = \mathrm{Pr}^{-1}(Y).$$

Hence, $\mathbf{x}' \in \mathrm{Pr}^{-1}(Y) \cap \mathrm{bd}(K) = W_Y$. Therefore, it has been shown that W_Y is symmetric about the closed line segment Y.

Second, it will be shown that W_Y is compact.

Recall that $\mathrm{Pr}^{-1}(Y) = Y + \{\lambda \mathbf{e}_3 \mid \lambda \in \mathbb{R}\}$. Lines are affine, so it follows from ***(vi)*** of Theorem 2.5.1 that $\{\lambda \mathbf{e}_3 \mid \lambda \in \mathbb{R}\}$ is closed.

The closed line segment Y is closed in $\mathbb{E}^3$. Let $\mathbf{y}_1, \mathbf{y}_2 \in Y \subseteq \mathrm{relbd}(B) \subseteq B \subseteq K$ be arbitrarily chosen. Recall that K is a convex body, which implies that K is bounded. It follows that there exists $M \in \mathbb{R}$ such that $\|\mathbf{y}_1 - \mathbf{y}_2\| < M$. Therefore, Y is bounded.

It follows by Theorem 2.8.4 that $Y + \{\lambda \mathbf{e}_3 \mid \lambda \in \mathbb{R}\} = \mathrm{Pr}^{-1}(Y)$ is closed.

By definition, $\mathrm{bd}(K) = \mathrm{cl}(K) \cap \mathrm{cl}(\mathbb{E}^n \backslash K)$. Recall that the closure of any set is closed. It follows by ***(ii)*** from Theorem 2.5.1 that $\mathrm{bd}(K)$ is closed.

Apply ***(ii)*** from Theorem 2.5.1 again to get that $W_Y = \mathrm{Pr}^{-1}(Y) \cap \mathrm{bd}(K)$ is closed.

Let $\mathbf{w}_1, \mathbf{w}_2 \in W_Y$ be arbitrarily chosen. It follows that $\mathbf{w}_1, \mathbf{w}_2 \in \mathrm{bd}(K) \subseteq K$. It follows that $\|\mathbf{w}_1 - \mathbf{w}_2\| < M$. This means that W_Y is bounded.

Hence, W_Y is compact.

Finally, it will be shown that W_Y is convex.

Let $\mathbf{x}, \mathbf{z} \in W_Y$ and $0 \leqslant \mu \leqslant 1$ be arbitrarily chosen. It follows that $\mathrm{Pr}(\mathbf{x}), \mathrm{Pr}(\mathbf{z}) \in Y$. Recall that line segments are convex; therefore, Y is convex. This means that $\mu \mathrm{Pr}(\mathbf{x}) + (1-\mu)\mathrm{Pr}(\mathbf{z}) \in Y$.Since W_Y is symmetric about Y, there exist $\tilde{\mu}, \hat{\mu} \in \mathbb{R}$ such that

$$\mathrm{Pr}(\mathbf{x}) = \mathbf{x} + \tilde{\mu}\mathbf{e}_3 \qquad \text{and} \qquad \mathrm{Pr}(\mathbf{z}) = \mathbf{z} + \hat{\mu}\mathbf{e}_3.$$

Then,

$$\mu\mathbf{x} + (1-\mu)\mathbf{z} = \mu\big(\mathrm{Pr}(\mathbf{x}) - \tilde{\mu}\mathbf{e}_3\big) + (1-\mu)\big(\mathrm{Pr}(\mathbf{z}) - \hat{\mu}\mathbf{e}_3$$
$$= \mu\mathrm{Pr}(\mathbf{x}) + (1-\mu)\mathrm{Pr}(\mathbf{z}) + (-\mu - \tilde{\mu}\mathbf{e}_3 - \mu\hat{\mu}\mathbf{e}_3)\,\mathbf{e}_3$$

$$\in Y + \{\lambda \mathbf{e}_3 \mid \lambda \in \mathbb{R}\} = \Pr^{-1}(Y).$$

Since K is convex and $\mathbf{x}, \mathbf{z} \in W_Y \subseteq \operatorname{bd}(K) \subseteq K$,it follows that $\mu\mathbf{x} + (1-\mu)\mathbf{z} \in K$. Again, recall that $K = \operatorname{int}(K) \cup \operatorname{bd}(K)$ where the sets $\operatorname{int}(K)$ and $\operatorname{bd}(K)$ are disjoint. It follows that either $\mu\mathbf{x} + (1-\mu)\mathbf{z} \in \operatorname{int}(K)$ or $\mu\mathbf{x} + (1-\mu)\mathbf{z} \in \operatorname{bd}(K)$.

Suppose for a contradiction that $\mu\mathbf{x} + (1-\mu)\mathbf{z} \in \operatorname{int}(K)$.

Since $\mu\mathbf{x} + (1-\mu)\mathbf{z} \in \Pr^{-1}(Y)$, it follows that $\Pr(\mu\mathbf{x} + (1-\mu)\mathbf{z}) \in Y$. Recall that $Y \subseteq \operatorname{relbd}(B) \subseteq \operatorname{bd}(K)$. Also, recall that K is affine plane symmetric. In particular, this means that there exists $\mathbf{w} \in K$ such that

$$\frac{1}{2}(\mathbf{w} + \mu\mathbf{x} + (1-\mu)\mathbf{z}) = \Pr(\mu\mathbf{x} + (1-\mu)\mathbf{z}).$$

It follows from Corollary 2.10.11 that

$$[\mu\mathbf{x} + (1-\mu)\mathbf{z}, \mathbf{w}) \subseteq \operatorname{int}(K).$$

Together, these imply that $\Pr(\mu\mathbf{x} + (1-\mu)\mathbf{z})) \in [\mu\mathbf{x} + (1-\mu)\mathbf{z}, \mathbf{w}) \subseteq \operatorname{int}(K)$. But, this is a contradiction because $\Pr(\mu\mathbf{x} + (1-\mu)\mathbf{z}) \in \operatorname{bd}(K)$. Therefore, $\mu\mathbf{x} + (1-\mu)\mathbf{z} \in \operatorname{bd}(K)$. Hence, $\mu\mathbf{x} + (1-\mu)\mathbf{z} \in W_Y$, which implies that W_Y is convex. ■

Given an arbitrary set S, a hyperplane H is said to *support* S at the point $\mathbf{s}$ if the set S is completely contained in one of the closed halfspaces determined by the hyperplane H and if $\mathbf{s} \in H \cap \operatorname{cl}(S)$. Let $\mathbf{x} \in \operatorname{relbd}(B)$ and let ℓ be a supporting line of B at $\mathbf{x}$ in the x_1x_2-plane. Each element belonging to $\ell' \cap \operatorname{relbd}(B)$ is called an *antipode* of $\mathbf{x}$ where ℓ' is a supporting line of B, which is parallel to and distinct from ℓ. The *complete antipode* of $\mathbf{x}$, which we denote by $A(\mathbf{x})$, is the set of all antipodes of $\mathbf{x}$.

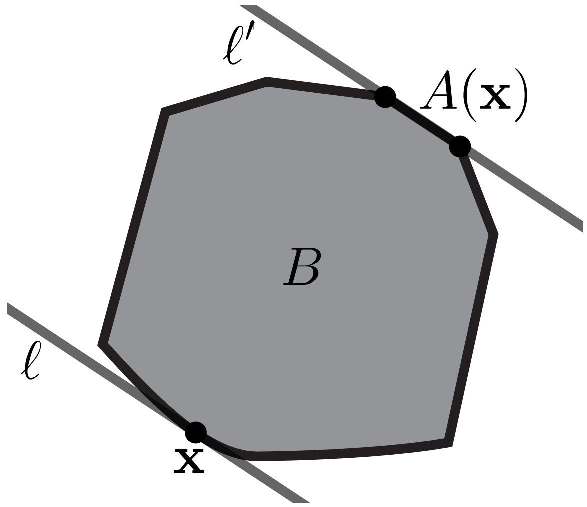


Figure 4.4

4.2 Proof of Theorem 4.1

4.2.1 Initial Observations

The following two results are useful in proving Theorem 4.1. In particular, Lemma 4.2.1.2 is required in sections 4.2.4, 4.2.7.3 and 4.2.8.3. The proof of Lemma 4.2.1.2 relies, in part, on Lemma 4.2.1.1. Moreover, Lemma 4.2.1.1 will be needed in 4.2.8.3.

Lemma 4.2.1.1. *Let* $\mathbf{k} \in \mathrm{relbd}(B)$ *be a cliff point with cliff* $[\mathbf{k}^-, \mathbf{k}^+]$*. Suppose that* $\widehat{\mathbf{d}} \in \mathbb{E}^3$ *is a direction with the property that* $\Pr(\widehat{\mathbf{d}})$ *illuminates* $\mathbf{k}$ *and* $\left\langle \widehat{\mathbf{d}}, \mathbf{e_3} \right\rangle < 0$*. Let* $\mathbf{p}_1 \in \mathbb{E}^3$ *such that* $r_{\widehat{\mathbf{d}}}^{\mathbf{k}^+} \cap \left(\mathbb{E}^2 \times \{0\}\right) = \{\mathbf{p}_1\}$*. Let* $\mathbf{p}_2 \in \mathbb{E}^3$ *be chosen such that* $\mathbf{k}$ *is the midpoint of the line segment* $[\mathbf{p}_1, \mathbf{p}_2]$*. Namely,* $\mathbf{p}_2 = \mathbf{p}_1 + 2(\mathbf{k} - \mathbf{p}_1)$*. Note that by convexity* $\mathbf{p}_2 \notin B$*.*

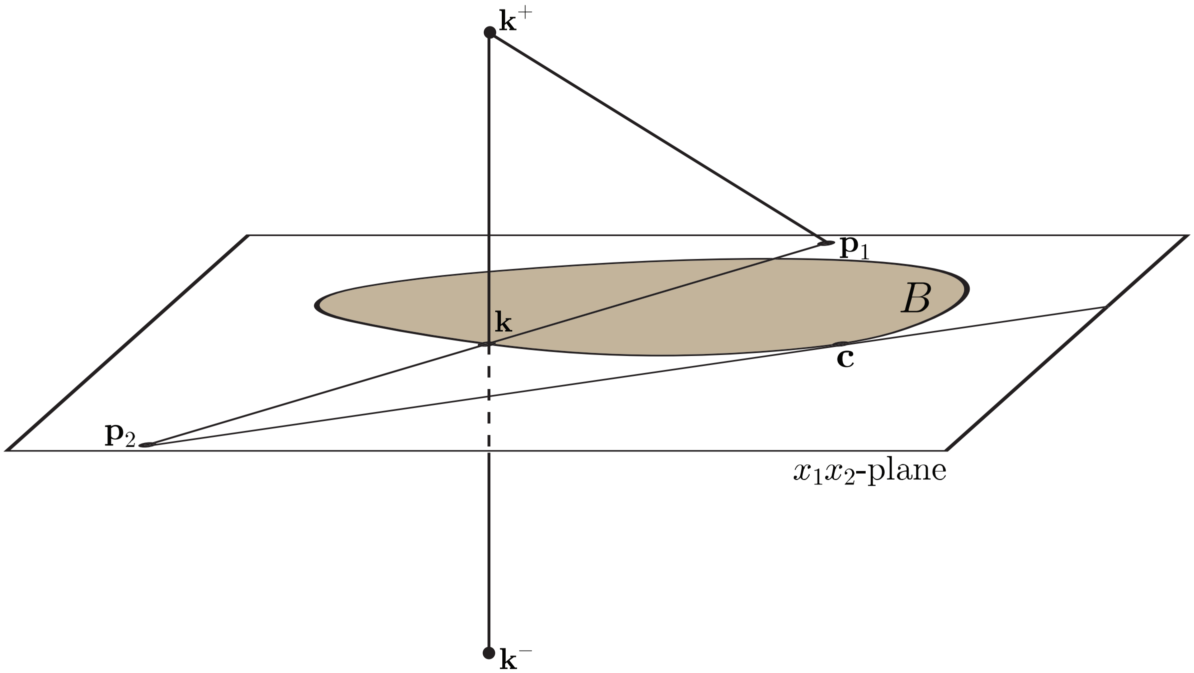


Figure 4.5

Also, note that the two supporting lines of B *through* $\mathbf{p_2}$ *may support* B *at more than one point. Let the supporting line* ℓ *and* $\mathbf{c} \in \ell \cap \mathrm{relbd}(B)$ *be chosen such that the orientation of* $\mathrm{conv}\{\mathbf{k}, \mathbf{p}_2, \mathbf{c}\}$ *is positive and* $\|\mathbf{c} - \mathbf{p}_2\| = \inf\{\|\mathbf{r} - \mathbf{p}_2\| \mid \mathbf{r} \in \ell \cap \mathrm{relbd}(B)\}$*. Then, every* $\mathbf{x} \in [\mathbf{k}, \mathbf{c})_B$ *is illuminated by* $\widehat{\mathbf{d}}$*.*

Proof. By assumption, $r^{\mathbf{k}}_{\mathrm{Pr}(\widehat{\mathbf{d}})} \cap \mathrm{int}(K) \neq \varnothing$. So, let $\mathbf{w} \in r^{\mathbf{k}}_{\mathrm{Pr}(\widehat{\mathbf{d}})} \cap \mathrm{int}(K)$. In other words, $\mathbf{w} = \mathbf{k} + \lambda \mathrm{Pr}(\widehat{\mathbf{d}}) \in \mathrm{int}(K)$ for some $\lambda > 0$.

It is useful to note that since $r^{\mathbf{k}^+}_{\widehat{\mathbf{d}}} \cap \left(\mathbb{E}^2 \times \{0\}\right) = \{\mathbf{p}_1\}$,

$$\mathbf{p}_1 = \mathrm{Pr}(\mathbf{p}_1) = \mathrm{Pr}(\mathbf{k}^+ + \varphi\widehat{\mathbf{d}}) = \mathbf{k} + \varphi \mathrm{Pr}(\widehat{\mathbf{d}}) \tag{4.3}$$

for some $\varphi > 0$. Also, $\langle \mathbf{k}^+, \mathbf{e_3} \rangle = -\varphi \left\langle \widehat{\mathbf{d}}, \mathbf{e_3} \right\rangle$, which implies $\langle \mathbf{k}^-, \mathbf{e_3} \rangle = \varphi \left\langle \widehat{\mathbf{d}}, \mathbf{e_3} \right\rangle$. Observe that by re-arranging (4.3), $\mathrm{Pr}(\widehat{\mathbf{d}}) = \dfrac{1}{\varphi}(\mathbf{p}_1 - \mathbf{k}) = \dfrac{1}{\varphi}(\mathbf{k} - \mathbf{p}_2)$.

It is also useful to note that $\widehat{\mathbf{d}} = \sigma(\mathbf{p}_1 - \mathbf{k}^+)$ for some $\sigma > 0$. Furthermore, $\mathbf{p}_1 - \mathbf{k}^+ = \mathbf{k}^- - \mathbf{p}_2$ since $\mathrm{conv}\{\mathbf{k}^+, \mathbf{p}_1, \mathbf{k}^-, \mathbf{p}_2\}$ is a parallelogram.

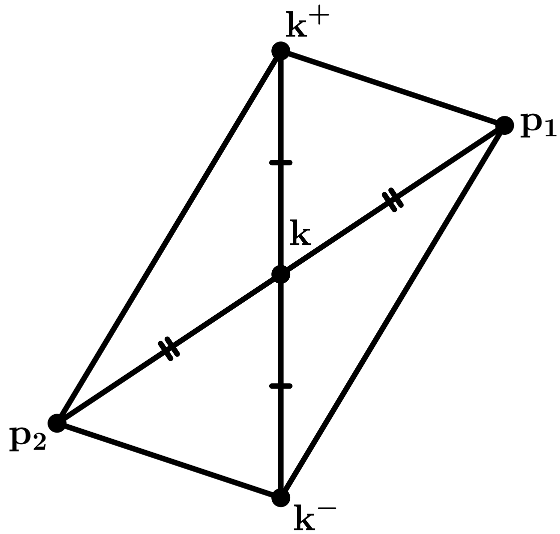


Figure 4.6: The diagonals bisect each other, therefore conv$\{\mathbf{k}^+, \mathbf{p}_1, \mathbf{k}^-, \mathbf{p}_2\}$ is a parallelogram.

Suppose $\mathbf{x} = \mathbf{k}$.

Since $\mathbf{k}^- \in [\mathbf{k}^-, \mathbf{k}^+] \subseteq \text{bd}(K) \subseteq K$, Lemma 1.1.8 in [51] implies that $(\mathbf{k}^-, \mathbf{w}] \in \text{int}(K)$.

Claim: The ray $r^{\mathbf{k}}_{\hat{\mathbf{d}}}$ intersects the line segment $(\mathbf{k}^-, \mathbf{w})$. In other words, $\mu\mathbf{k}^- + (1-\mu)\mathbf{w} \in r^{\mathbf{k}}_{\hat{\mathbf{d}}}$ for some $0 < \mu < 1$.

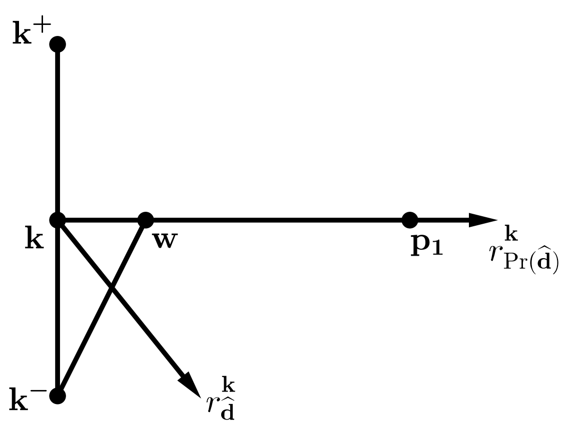


Figure 4.7

Let $\mu = \dfrac{\lambda}{\lambda + \varphi}$. Since $\lambda, \varphi > 0$, it follows that $\mu > 0$. Furthermore, $\lambda + \varphi > \lambda$. Therefore, it also follows that $\mu < 1$. Consider the following,

$$\begin{aligned}
\mu\mathbf{k}^- + (1-\mu)\mathbf{w} &= \mu\mathbf{k}^- + (1-\mu)(\mathbf{k} + \lambda\mathrm{Pr}(\widehat{\mathbf{d}})) \\
&= \mathbf{k} + \lambda(1-\mu)\mathrm{Pr}(\widehat{\mathbf{d}}) + \mu(\mathbf{k}^- - \mathbf{k}) \\
&= \mathbf{k} + \lambda(1-\mu)\mathrm{Pr}(\widehat{\mathbf{d}}) + \mu\left\langle \mathbf{k}^-, \mathbf{e_3}\right\rangle \mathbf{e_3} \\
&= \mathbf{k} + \lambda(1-\mu)\mathrm{Pr}(\widehat{\mathbf{d}}) + \mu\varphi\left\langle \widehat{\mathbf{d}}, \mathbf{e_3}\right\rangle \mathbf{e_3} \\
&= \mathbf{k} + \lambda\left(\frac{\varphi}{\lambda+\varphi}\right)\mathrm{Pr}(\widehat{\mathbf{d}}) + \left(\frac{\lambda\varphi}{\lambda+\varphi}\right)\left\langle \widehat{\mathbf{d}}, \mathbf{e_3}\right\rangle \mathbf{e_3} \\
&= \mathbf{k} + \varphi\left(\frac{\lambda}{\lambda+\varphi}\right)\widehat{\mathbf{d}}.
\end{aligned}$$

Hence, $\mu\mathbf{k}^- + (1-\mu)\mathbf{w} \in (\mathbf{k}^-, \mathbf{w}) \cap r^{\mathbf{k}}_{\widehat{\mathbf{d}}}$. In other words, $\mu\mathbf{k}^- + (1-\mu)\mathbf{w} \in \mathrm{int}(K) \cap r^{\mathbf{k}}_{\widehat{\mathbf{d}}}$. Thus, $\widehat{\mathbf{d}}$ illuminates $\mathbf{k}$.

Let $\mathbf{x} \in (\mathbf{k}, \mathbf{c})_B$ be arbitrarily chosen.

Case 1: Suppose that $[\mathbf{k}, \mathbf{c}]_B$ is a side of B. In other words, suppose that $[\mathbf{k}, \mathbf{c}]_B = [\mathbf{k}, \mathbf{c}]$. This means $\mathbf{x} \in (\mathbf{k}, \mathbf{c})$.

<u>Claim:</u> $\mathrm{conv}\{\mathbf{k}^-, \mathbf{k}^+, \mathbf{c}\} \subseteq \mathrm{bd}(K)$.

Note that $\mathbf{k} = \frac{1}{2}\mathbf{k}^- + \frac{1}{2}\mathbf{k}^+ + 0\,\mathbf{c}$, which means $\mathbf{k} \in \mathrm{conv}\{\mathbf{k}^-, \mathbf{k}^+, \mathbf{c}\}$. Thus,

$$[\mathbf{k}, \mathbf{c}] \subseteq \mathrm{conv}\{\mathbf{k}^-, \mathbf{k}^+, \mathbf{c}\}.$$

Note that by Proposition 4.1.3, $W_{[\mathbf{k},\mathbf{c}]}$ is a compact convex set in the plane $\mathrm{Pr}^{-1}([\mathbf{k}, \mathbf{c}])$. Also, notice that $\mathbf{k}^-, \mathbf{k}^+, \mathbf{c} \in W_{[\mathbf{k},\mathbf{c}]}$. These two facts together imply that $\mathrm{conv}\{\mathbf{k}^-, \mathbf{k}^+, \mathbf{c}\} \subseteq W_{[\mathbf{k},\mathbf{c}]}$. By definition, $W_{[\mathbf{k},\mathbf{c}]} \subseteq \mathrm{bd}(K)$. Hence, $\mathrm{conv}\{\mathbf{k}^-, \mathbf{k}^+, \mathbf{c}\} \subseteq \mathrm{bd}(K)$.

Note that $\mathbf{x} = \eta'\mathbf{k} + (1-\eta)\mathbf{c}$ for some $0 < \eta' < 1$. Let $\mathbf{f} = \eta'\mathbf{k}^+ + (1-\eta')\mathbf{c}$ and $\mathbf{e} = \eta'\mathbf{k}^- + (1-\eta')\mathbf{c}$. As defined, $\mathbf{x} = \frac{1}{2}\mathbf{f} + \frac{1}{2}\mathbf{e}$.

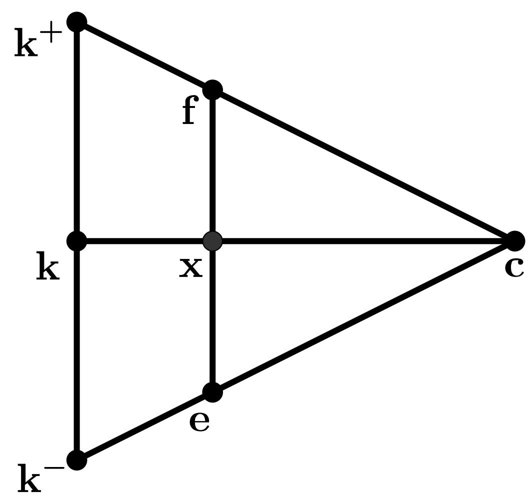


Figure 4.8

Recall from above that $\mathbf{w} = \mathbf{k} + \lambda \mathrm{Pr}(\widehat{\mathbf{d}}) \in \mathrm{int}(k)$ for some $\lambda > 0$. Applying Lemma 1.1.8 in [51] again, results in $[\mathbf{w}, \mathbf{c}) \subseteq \mathrm{int}(K)$. Observe that

$$\begin{aligned} \eta'\mathbf{w} + (1-\eta')\mathbf{c} &= \eta'(\mathbf{k} + \lambda \mathrm{Pr}(\widehat{\mathbf{d}})) + (1-\eta')\mathbf{c} \\ &= \eta'\mathbf{k} + (1-\eta')\mathbf{c} + \eta'\lambda \mathrm{Pr}(\widehat{\mathbf{d}}) \\ &= \mathbf{x} + \eta'\lambda \mathrm{Pr}(\widehat{\mathbf{d}}). \end{aligned}$$

where $\eta'\lambda > 0$. This implies that $\eta'\mathbf{w} + (1-\eta')\mathbf{c} \in [\mathbf{w}, \mathbf{c}) \cap r^{\mathbf{x}}_{\mathrm{Pr}(\widehat{\mathbf{d}})}$. In other words, $\eta'\mathbf{w} + (1-\eta')\mathbf{c} \in \mathrm{int}(K) \cap r^{\mathbf{x}}_{\mathrm{Pr}(\widehat{\mathbf{d}})}$. Using Lemma 1.1.8 in [51] yet again, results in $(\mathbf{e}, \eta'\mathbf{w} + (1-\eta')\mathbf{c}] \in \mathrm{int}(K)$. The same method used in the first part of the proof will be used now to show $r^{\mathbf{x}}_{\widehat{\mathbf{d}}} \cap (\mathbf{e}, \eta'\mathbf{w} + (1-\eta')\mathbf{c}] \neq \varnothing$.

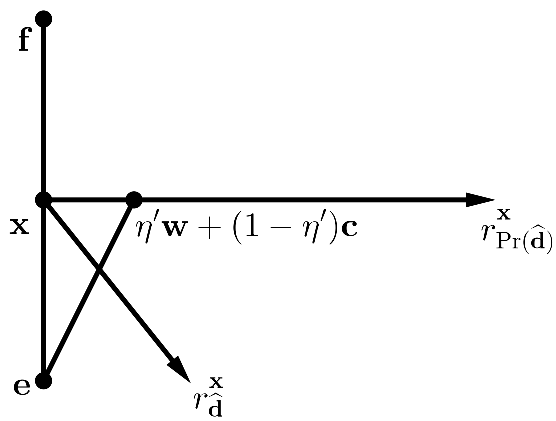


Figure 4.9

Recall that $\langle \mathbf{k}^-, \mathbf{e_3} \rangle = \varphi \left\langle \widehat{\mathbf{d}}, \mathbf{e_3} \right\rangle$ for some $\varphi > 0$. Note that $\|\mathbf{k}^- - \mathbf{k}\| < \|\mathbf{e} - \mathbf{x}\|$ or in other words, $(\mathbf{e} - \mathbf{f}) = \gamma'(\mathbf{k}^- - \mathbf{k})$ where $0 < \gamma' < 1$. Let $\xi' = \dfrac{\eta'\lambda}{\gamma'\varphi + \eta'\lambda}$. Since $\eta'\lambda > 0$ and $\gamma'\varphi > 0$, $\xi' > 0$. Furthermore, $\gamma'\varphi + \eta'\lambda > \eta'\lambda$, which implies $\xi' < 1$. Consider the following,

$$\begin{aligned}
\xi'\mathbf{e} - (1-\xi')[\eta'\mathbf{w} + (1-\eta')\mathbf{c}] &= \xi'\mathbf{e} + (1-\xi')[\mathbf{x} + \eta'\lambda \mathrm{Pr}(\widehat{\mathbf{d}})] \\
&= \mathbf{x} + \xi'(\mathbf{e} - \mathbf{x}) + (1-\xi')\eta'\lambda \mathrm{Pr}(\widehat{\mathbf{d}}) \\
&= \mathbf{x} + \xi'\gamma'(\mathbf{k}^- - \mathbf{k}) + (1-\xi')\eta'\lambda \mathrm{Pr}(\widehat{\mathbf{d}}) \\
&= \mathbf{x} + \xi'\gamma' \left\langle \mathbf{k}^-, \mathbf{e_3} \right\rangle \mathbf{e_3} + (1-\xi')\eta'\lambda \mathrm{Pr}(\widehat{\mathbf{d}}) \\
&= \mathbf{x} + \xi'\gamma'\varphi \left\langle \widehat{\mathbf{d}}, \mathbf{e_3} \right\rangle \mathbf{e_3} + (1-\xi')\eta'\lambda \mathrm{Pr}(\widehat{\mathbf{d}}) \\
&= \mathbf{x} + \frac{\eta'\lambda}{\gamma'\varphi + \eta'\lambda}\widehat{\mathbf{d}}
\end{aligned}$$

Hence, $\xi'\mathbf{e} - (1-\xi')[\eta'\mathbf{w} + (1-\eta')\mathbf{c}] \in r^{\mathbf{x}}\widehat{\mathbf{d}} \cap (\mathbf{e}, \eta'\mathbf{w} + (1-\eta')\mathbf{c}]$. This means that $r^{\mathbf{x}}\widehat{\mathbf{d}} \cap \mathrm{int}(K) \neq \varnothing$. Thus, $\widehat{\mathbf{d}}$ illuminates $\mathbf{x}$.

Case 2: Suppose that $[\mathbf{k}, \mathbf{c}]_B$ is not a side of B. Namely, suppose that $[\mathbf{k}, \mathbf{c}]_B \neq [\mathbf{k}, \mathbf{c}]$. Note that $\mathbf{x} \in \mathrm{conv}\{\mathbf{k}, \mathbf{p}_2, \mathbf{c}\}$. This means that there exists $0 < \sigma < 1$ and $0 < \sigma' < 1 - \sigma$ such that $\mathbf{x} = \sigma\mathbf{k} + \sigma'\mathbf{c} + (1 - \sigma - \sigma')\mathbf{p}_2$.

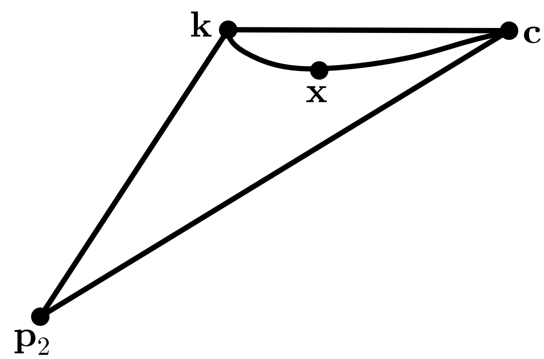


Figure 4.10

Claim: $r^{x}_{\Pr(\hat{\mathbf{d}})} \cap (\mathbf{k}, \mathbf{c}) \neq \varnothing$.

Notice that $(1 - \sigma - \sigma') \cdot \varphi > 0$. Now, observe that

$$\begin{aligned}
\mathbf{x} + [(1 - \sigma - \sigma') \cdot \varphi] \Pr(\widehat{\mathbf{d}}) &= \sigma\mathbf{k} + \sigma'\mathbf{c} + (1 - \sigma - \sigma')\mathbf{p}_2 + [(1 - \sigma - \sigma') \cdot \varphi] \Pr(\widehat{\mathbf{d}}) \\
&= \sigma\mathbf{k} + \sigma'\mathbf{c} + (1 - \sigma - \sigma')\mathbf{p}_2 + (1 - \sigma - \sigma')(\mathbf{k} - \mathbf{p}_2) \\
&= \sigma\mathbf{k} + \sigma'\mathbf{c} + (1 - \sigma - \sigma')\mathbf{k} \\
&= (1 - \sigma')\mathbf{k} + \sigma'\mathbf{c}.
\end{aligned}$$

Since $0 < \sigma' < 1 - \sigma < 1$, $(1 - \sigma')\mathbf{k} + \sigma'\mathbf{c} \in r^{\mathbf{x}}_{\Pr(\hat{\mathbf{d}})} \cap (\mathbf{k}, \mathbf{c})$.

Let $\mathbf{f} = (1 - \sigma')\mathbf{k}^{+} + \sigma'\mathbf{c}$ and $\mathbf{e} = (1 - \sigma')\mathbf{k}^{-} + \sigma'\mathbf{c}$. As defined, $(1 - \sigma')\mathbf{k} + \sigma'\mathbf{c} = \frac{1}{2}\mathbf{f} + \frac{1}{2}\mathbf{e}$.

Claim: $(\mathbf{e}, \mathbf{f}) \subseteq \operatorname{int}(K)$.

First, it will be shown that $(1 - \sigma')\mathbf{k} + \sigma'\mathbf{c} \in \operatorname{int}(K)$.

The supposition that $[\mathbf{a}, \mathbf{b}]$ is not a side of B means that $[\mathbf{a}, \mathbf{b}] \nsubseteq \operatorname{relbd}(B)$. Since $\mathbf{a}, \mathbf{b} \in \operatorname{relbd}(B)$ by assumption, it follows that $(\mathbf{a}, \mathbf{b}) \nsubseteq \operatorname{relbd}(B)$. Since B is convex, the line segment $[\mathbf{a}, \mathbf{b}]$ is contained in B. This implies that $(\mathbf{a}, \mathbf{b}) \subseteq \operatorname{relint}(B)$ due to the fact that $B = \operatorname{relbd}(B) \cup \operatorname{relint}(B)$ where $\operatorname{relbd}(B) \cap \operatorname{relint}(B) = \varnothing$. Thus, $(1 - \sigma')\mathbf{a} - \sigma'\mathbf{b} \in \operatorname{relint}(B) \subseteq \operatorname{int}(K)$.

Since $\mathbf{a}^{-}, \mathbf{a}^{+}, \mathbf{b} \in \operatorname{bd}(K) \subseteq K$ and K is convex, it follows that $[\mathbf{a}^{-}, \mathbf{b}] \subseteq K$ and $[\mathbf{a}^{+}, \mathbf{b}] \subseteq K$.

Thus, $\mathbf{e} = (1-\sigma')\mathbf{a}^- + \sigma'\mathbf{b} \in K$ and $\mathbf{f} = (1-\sigma')\mathbf{a}^+ + \sigma'\mathbf{b} \in K$. By Lemma 1.1.8 in [51], $(\mathbf{f}, (1-\sigma')\mathbf{a}+\sigma'\mathbf{b}] \subseteq \text{int}(K)$ and $[(1-\sigma')\mathbf{a}+\sigma'\mathbf{b}, \mathbf{e}) \subseteq \text{int}(K)$. Hence, $(\mathbf{f}, (1-\sigma')\mathbf{a}+\sigma'\mathbf{b}] \cup [(1-\sigma')\mathbf{a}+\sigma'\mathbf{b}, \mathbf{e}) = (\mathbf{f}, \mathbf{e}) = (\mathbf{e}, \mathbf{f}) \subseteq \text{int}(K)$.

Since $1-\sigma-\sigma' > 0$,

$$\begin{aligned}
\mathbf{x} + (1-\sigma-\sigma')\widehat{\mathbf{d}} &= \sigma\mathbf{k} + \sigma'\mathbf{c} + (1-\sigma-\sigma')\mathbf{p}_2 + (1-\sigma-\sigma')(\mathbf{k}^- - \mathbf{p}_2) \\
&= \sigma\mathbf{k} + \sigma'\mathbf{c} + (1-\sigma-\sigma')\mathbf{k}^- \\
&= \left(\frac{\sigma}{1-\sigma'}\right)[(1-\sigma')\mathbf{k} + \sigma'\mathbf{c}] + \left(\frac{1-\sigma-\sigma'}{1-\sigma'}\right)[(1-\sigma')\mathbf{k}^- + \sigma'\mathbf{c}] \\
&= \left(\frac{\sigma}{1-\sigma'}\right)[(1-\sigma')\mathbf{k} + \sigma'\mathbf{c}] + \left(1-\frac{\sigma}{1-\sigma'}\right)[(1-\sigma')\mathbf{k}^- + \sigma'\mathbf{c}] \\
&\in r^{\mathbf{x}}_{\widehat{\mathbf{d}}} \cap ((1-\sigma')\mathbf{k} + \sigma'\mathbf{c}, \mathbf{e}).
\end{aligned}$$

Hence, $\widehat{\mathbf{d}}$ illuminates $\mathbf{x}$. ■

Lemma 4.2.1.2. *Let* $[\mathbf{u}, \mathbf{v}] \subseteq \text{relbd}(B)$ *be a side and suppose that* $\text{relint}(B)$ *contains a segment* $[\mathbf{n}, \mathbf{m}]$ *such that* $\mathbf{m}-\mathbf{n} = 2(\mathbf{v}-\mathbf{u})$. *Then* $W_{[\mathbf{u},\mathbf{v}]}$ *can be illuminated by two directions.*

Proof. Let $\mathbf{m}$ and $\mathbf{n}$ denote the endpoints of the line segment completely contained in $\text{relint}(B)$ which is twice as long and parallel to the side $[\mathbf{u}, \mathbf{v}] \subseteq \text{relbd}(B)$ where the points $\mathbf{u}, \mathbf{v}, \mathbf{m}$ and $\mathbf{n}$ follow each other in this order when starting at the point $\mathbf{u}$ and travelling counter-clockwise.

Suppose $W_{[\mathbf{u},\mathbf{v}]} = [\mathbf{u}, \mathbf{v}]$.

Claim: The directions $\mathbf{d_1} = \frac{1}{2}(\mathbf{n}+\mathbf{m}) - \mathbf{u}$ and $\mathbf{d_2} = \frac{1}{2}(\mathbf{n}+\mathbf{m}) - \mathbf{v}$ illuminate the entire line segment $[\mathbf{u}, \mathbf{v}]$.

Let $\mathbf{z} \in [\mathbf{u}, \mathbf{v}]$ be arbitrarily chosen.If $\mathbf{z} \in [\mathbf{u}, \frac{1}{2}(\mathbf{u}+\mathbf{v})]$, then $\mathbf{z} = \mu\left(\frac{1}{2}(\mathbf{u}+\mathbf{v})\right) + (1-\mu)\mathbf{u}$ for some $0 \leqslant \mu \leqslant 1$. Clearly,

$$\mathbf{z} + \mathbf{d_1} = \frac{\mu}{2}(\mathbf{u}+\mathbf{v}) + (1-\mu)\mathbf{u} + \frac{1}{2}(\mathbf{n}+\mathbf{m}) - \mathbf{u} \in r^{\mathbf{z}}_{\mathbf{d_1}}.$$

Moreover,

$$\mathbf{z} + \mathbf{d_1} = \frac{\mu}{2}(\mathbf{u}+\mathbf{v}) + (1-\mu)\mathbf{u} + \frac{1}{2}(\mathbf{n}+\mathbf{m}) - \mathbf{u}$$

$$= \frac{\mu}{2}(\mathbf{v}-\mathbf{u}) + \frac{1}{2}(\mathbf{n}+\mathbf{m})$$
$$= \frac{\mu}{4}(\mathbf{m}-\mathbf{n}) + \frac{1}{2}(\mathbf{n}+\mathbf{m})$$
$$= \left(\frac{1}{2}-\frac{\mu}{4}\right)\mathbf{n} + \left(\frac{1}{2}+\frac{\mu}{4}\right)\mathbf{m}.$$

Since $0 \leqslant \mu \leqslant 1$, it follows that $0 < \frac{1}{2} + \frac{\mu}{4} < 1$. Thus,

$$\left(\frac{1}{2}-\frac{\mu}{4}\right)\mathbf{n} + \left(\frac{1}{2}+\frac{\mu}{4}\right)\mathbf{m} \in [\mathbf{n},\mathbf{m}] \subseteq \text{int}B \subseteq \text{int}(K).$$

In other words, $r^{\mathbf{z}}_{\mathbf{d_1}} \cap \text{int}(K) \neq \varnothing$. So, $\mathbf{d_1}$ illuminates $\mathbf{z}$.

Similarly, if $\mathbf{z} \in [\frac{1}{2}(\mathbf{u}+\mathbf{v}), \mathbf{v}]$, then $\mathbf{z} = \mu\left(\frac{1}{2}(\mathbf{u}+\mathbf{v})\right) + (1-\mu)\mathbf{v}$ for some $0 \leqslant \mu \leqslant 1$ and

$$\left(\frac{1}{2}+\frac{\mu}{4}\right)\mathbf{n} + \left(\frac{1}{2}-\frac{\mu}{4}\right)\mathbf{m} = -\frac{\mu}{4}(\mathbf{m}-\mathbf{n}) + \frac{1}{2}(\mathbf{n}+\mathbf{m})$$
$$= -\frac{\mu}{2}(\mathbf{v}-\mathbf{u}) + \frac{1}{2}(\mathbf{n}+\mathbf{m}) + (1-1)\mathbf{v}$$
$$= \frac{\mu}{2}(\mathbf{u}+\mathbf{v}) + (1-\mu)\mathbf{v} + \frac{1}{2}(\mathbf{n}+\mathbf{m}) - \mathbf{v} \in r^{\mathbf{z}}_{\mathbf{d_1}} \cap \text{int}(K).$$

So, $\mathbf{d_2}$ illuminates $\mathbf{z}$.

Suppose now that $W_{[\mathbf{u},\mathbf{v}]} \neq [\mathbf{u},\mathbf{v}]$.This means that $[\mathbf{u},\mathbf{v}]$ contains cliff points. Let $\mathbf{k} \in [\mathbf{u},\mathbf{v}]$ be a cliff point chosen so that

$$\|\mathbf{k}^+ - \mathbf{k}^-\| = \max\{\|\mathbf{f}^+ - \mathbf{f}^-\| \mid \text{for all cliff points } \mathbf{f} \in [\mathbf{u},\mathbf{v}]\}.$$

Let $\mathbf{p}_1 \in [\mathbf{n},\mathbf{m}]$ be chosen such that $\mathbf{p}_1 - \mathbf{n} = 2(\mathbf{k}-\mathbf{u})$ and $\mathbf{m} - \mathbf{p}_1 = 2(\mathbf{v}-\mathbf{k})$.

Claim: The directions $\mathbf{d}^+ = (\mathbf{p}_1 - \mathbf{k}) - (\mathbf{k}^+ - \mathbf{k})$ and $\mathbf{d}^- = (\mathbf{p}_1 - \mathbf{k}) - (\mathbf{k}^- - \mathbf{k})$ illuminate $W_{[\mathbf{u},\mathbf{v}]}$.

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are cliff points. This means that all points in the interval $[\mathbf{u},\mathbf{v}]$ are cliff points, by convexity. Let $\mathbf{w} \in \left(W_{[\mathbf{u},\mathbf{v}]}\right)_+$ be arbitrarily chosen. Clearly, $\|\mathbf{w} - \text{Pr}(\mathbf{w})\| \leqslant \|\mathbf{k}^+ - \mathbf{k}\|$. Equivalently, $(\mathbf{k}^+ - \mathbf{k}) = (\mathbf{k} - \mathbf{k}^-) = \varphi(\mathbf{w} - \text{Pr}(\mathbf{w}))$ for some $\varphi \geqslant 1$. Also, note that $\text{Pr}(\mathbf{w}) = \gamma'\mathbf{u} + (1-\gamma')\mathbf{v}$ for some $0 \leqslant \gamma' \leqslant 1$.

Let $\lambda = \dfrac{1}{\varphi}$. Since $\varphi \geqslant 1$, it follows that $0 < \lambda \leqslant 1$. If $\mathbf{w} \in \left(W_{[\mathbf{u},\mathbf{v}]}\right)_+$, then

$$\mathbf{w} + \lambda(\mathbf{d}^+) = \mathbf{w} + \lambda[(\mathbf{p}_1 - \mathbf{k}) - (\mathbf{k}^+ - \mathbf{k})]$$

$$
\begin{aligned}
&= \mathbf{w} + \lambda\left[\mathbf{p}_1 - \frac{1}{2}(\mathbf{p}_1 - \mathbf{n}) - \mathbf{u} - \varphi(\mathbf{w} - \mathrm{Pr}(\mathbf{w}))\right] \\
&= \mathrm{Pr}(\mathbf{w}) - \lambda\mathbf{u} + \frac{\lambda}{2}(\mathbf{p}_1 + \mathbf{n}) \\
&= (1-\lambda)\mathrm{Pr}(\mathbf{w}) + \lambda\mathrm{Pr}(\mathbf{w}) - \lambda\mathbf{u} + \frac{\lambda}{2}(\mathbf{p}_1 + \mathbf{n}) \\
&= (1-\lambda)\mathrm{Pr}(\mathbf{w}) + \lambda(\gamma'\mathbf{u}(1-\gamma')\mathbf{v} - \mathbf{u}) + \frac{\lambda}{2}(\mathbf{p}_1 + \mathbf{n}) \\
&= (1-\lambda)\mathrm{Pr}(\mathbf{w}) + \lambda(1-\gamma')(\mathbf{v} - \mathbf{u}) + \frac{\lambda}{2}(\mathbf{p}_1 + \mathbf{n}) \\
&= (1-\lambda)\mathrm{Pr}(\mathbf{w}) + \frac{\lambda(1-\gamma')}{2}(\mathbf{m} - \mathbf{n}) + \frac{\lambda}{2}(\mathbf{p}_1 + \mathbf{n}) \\
&= (1-\lambda)\mathrm{Pr}(\mathbf{w}) + \frac{\lambda}{2}(\mathbf{p}_1 + \gamma'\mathbf{n} + (1-\gamma')\mathbf{m}) \\
&= (1-\lambda)\mathrm{Pr}(\mathbf{w}) + \frac{\lambda}{2}(\gamma'\mathbf{p}_1 + \gamma'\mathbf{n} + (1-\gamma')\mathbf{p}_1 + (1-\gamma')\mathbf{m}) \\
&= (1-\lambda)\mathrm{Pr}(\mathbf{w}) + \lambda\left(\frac{\gamma'}{2}(\mathbf{p}_1 + \mathbf{n}) + \frac{(1-\gamma')}{2}(\mathbf{p}_1 + \mathbf{m})\right) \in r^{\mathbf{w}}_{\mathbf{d}^+} \cap \mathrm{int}(K).
\end{aligned}
$$

Namely, it has been shown that the direction $\mathbf{d}^+$ illuminates $\mathbf{w}$.

If $\mathbf{w} \in \left(W_{[\mathbf{u},\mathbf{v}]}\right)_-$, the exact same procedure shows that $\mathbf{w}$ is illuminated by the direction $\mathbf{d}^-$.

Now, suppose that either $\mathbf{u}$ or $\mathbf{v}$ is a ground point. Without loss of generality, let $\mathbf{u}$ be a ground point. Also, let $\mathbf{p}_2 = \mathbf{p}_1 + 2(\mathbf{k} - \mathbf{p}_1)$.

Sub-claim: The points $\mathbf{p}_2$, $\mathbf{u}$ and $\mathbf{n}$ are collinear.

Consider the line through the points $\mathbf{p}_2$ and $\mathbf{n}$, $\lambda\mathbf{p}_2 + (1-\lambda)\mathbf{n}$. When $\lambda = \frac{1}{2}$, the result is $\mathbf{k} - \frac{1}{2}(\mathbf{p}_1 - \mathbf{n}) = \mathbf{u}$, which shows $\mathbf{p}_2$, $\mathbf{u}$ and $\mathbf{n}$ are collinear.

Note that since $\mathbf{u} \in \mathrm{relbd}(B)$ and $\mathbf{n} \in \mathrm{relint}(B)$, $\mathbf{p}_2 \notin B$ by convexity. Let ℓ be a supporting line of B passing through the point $\mathbf{p}_2$ such that $\mathbf{u} \in [\mathbf{k}, \mathbf{c})_B$ where $\mathbf{c} \in \ell \cap \mathrm{relbd}(B)$ and $\|\mathbf{c}-\mathbf{p}_2\| = \inf\{||\mathbf{s}-\mathbf{p}_2\| \mid \mathbf{s} \in \ell \cap \mathrm{relbd}(B)\}$. Such a $\mathbf{c} \in \mathrm{relbd}(B)$ exists since $[\mathbf{n}, \mathbf{m}] \in \mathrm{relint}(B)$. Notice that $\mathrm{conv}\{\mathbf{k}, \mathbf{p}_2, \mathbf{c}\}$ has positive orientation. Furthermore, $\mathrm{Pr}(\mathbf{d}^+) = \mathbf{p}_1 - \mathbf{k}$ illuminates $\mathbf{k}$ since $\mathbf{k} + \mathrm{Pr}(\mathbf{d}^+) = \mathbf{k} + (\mathbf{p}_1 - \mathbf{k}) = \mathbf{p}_1 \in r^{\mathbf{d}^+}_{\mathbf{k}} \cap \mathrm{int}(K)$. Thus, $\mathbf{d}^+$ illuminates $\mathbf{u}$ by Lemma 4.2.1.1. ∎

4.2.2 First Major Case of Theorem 4.1

In this case, suppose that $\mathrm{relbd}(B)$ has no sides.

Note that by Theorem 2.2.4 in [51], there are only countably many singular points in the boundary of K and since $\text{relbd}(B) \subseteq \text{bd}(K)$, there are only countably many singular points in $\text{relbd}(B)$. In other words, smooth points are dense in $\text{bd}(K)$ and $\text{relbd}(B)$.

Let $\mathbf{p} \in \text{relbd}(B)$ be an arbitrarily chosen smooth point. By definition, the supporting line of B in the x_1x_2-plane at $\mathbf{p}$ is unique. Denote this supporting line by ℓ. Since B is a convex body in the x_1x_2-plane, it follows from Theorem 6 in [19] that there exists exactly one other distinct supporting line of B in the x_1x_2-plane, parallel to ℓ. Call this supporting line ℓ'. Since B has no sides, it follows that the supporting line ℓ' supports B at a single point, $\mathbf{q}$. Thus, the complete antipode of $\mathbf{p}$ contains only a single point. Namely, $A(\mathbf{p}) = \{\mathbf{q}\}$.

Either the point $\mathbf{q}$ is a ground point or it is a cliff point. In examining these two possibilities, the first major case of Theorem 4.1 will be split into the sub-cases 4.2.2.1 and 4.2.2.2.

4.2.2.1 Suppose that q is a ground point.

A summary of the proof for this sub-case is described here with an explanation of how each Proposition and Lemma fit together to make the proof. First, it is shown that $\mathbf{q}$ is illuminated by the direction $\mathbf{p}-\mathbf{q}$. The proof of Proposition 4.2.2.1.1 does not make use of the fact that $\text{relbd}(B)$ has no sides and therefore, can be used again in Proposition 4.2.3.1. Second, an open set on the boundary of K containing $\mathbf{q}$ is found and it is shown in Proposition 4.2.2.1.2 that every element of this open set is also illuminated by the direction $\mathbf{p}-\mathbf{q}$. The open set on the boundary of K containing $\mathbf{q}$ is denoted by $U(\mathbf{q})$. Lemma Lemma 4.2.2.1.3 shows that there exists a closed set slab $\left[\text{Pr}^{-1}(\ell_*), \text{Pr}^{-1}(\ell')\right] \cap \text{bd}(K)$, for some line ℓ_* strictly between and parallel to the supporting lines ℓ and ℓ', that is contained by $U(\mathbf{q})$. Then, Propositions 4.2.2.1.8 and 4.2.2.1.9 show that there exist two points, $\mathbf{a}$ and $\mathbf{b}$, of $\text{relbd}(B)$ that belong to $U(\mathbf{q})$ and that lie on the line ℓ_*. Finally, Lemma Lemma 4.2.2.1.12 shows that the remaining part of the boundary, $\text{bd}(K)\backslash U(\mathbf{q})$, is illuminated by the six directions $\frac{1}{2}(\mathbf{q}+\mathbf{a})-\mathbf{b}$, $\frac{1}{2}(\mathbf{q}+\mathbf{b})-\mathbf{a}$, $\left(1-\frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{a})-\mathbf{b}\right) \pm \mathcal{T}\mathbf{e}_3$ and $\left(1-\frac{2(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{b})-\mathbf{a}\right) \pm \mathcal{T}\mathbf{e}_3$.

Note that Lemma Lemma 4.2.2.1.12 never makes use of the fact that relbd(B) has no sides and therefore, will be used in all other cases with the points $\mathbf{p}$ and $\mathbf{q}$ carefully chosen.

Proposition 4.2.2.1.1. *The point* $\mathbf{q}$ *is illuminated by the direction* $\mathbf{p} - \mathbf{q}$.

Proof. Recall from above that $\mathbf{q} \in \ell'$. It follows from the fact that the line ℓ' is parallel to but distinct from the line ℓ that there exists some $\mathbf{t} \neq \mathbf{o}$ such that $\ell' = \ell + \mathbf{t}$. Since B is convex, $[\mathbf{p}, \mathbf{q}] \subseteq B$. Suppose for a contradiction that $[\mathbf{p}, \mathbf{q}] \subseteq \text{relbd}(B)$. It follows from Theorem 14 in [39] that there exists a supporting line of B which contains the closed interval $[\mathbf{p}, \mathbf{q}]$. Since $\mathbf{p}$ is a smooth point, the support line ℓ is unique. This means $[\mathbf{p}, \mathbf{q}] \subseteq \ell$. Therefore, $\mathbf{q} \in \ell$. This is a contradiction. Hence, $(\mathbf{p}, \mathbf{q}) \nsubseteq \text{relbd}(B)$, since $\mathbf{p}, \mathbf{q} \in \text{relbd}(B)$. Recall from Properties 4.1.1 that B is closed and its relative interior is non-empty in the x_1x_2-plane. It follows that $B = \text{relint}(B) \cup \text{relbd}(B)$. Also, recall that the relative interior and the relative boundary of any set are disjoint. Thus, $(\mathbf{p}, \mathbf{q}) \subseteq \text{relint}(B) \subseteq \text{int}(K)$. So, $\frac{1}{2}(\mathbf{p} + \mathbf{q}) = \mathbf{q} + \frac{1}{2}(\mathbf{p} - \mathbf{q}) \in r^{\mathbf{q}}_{\mathbf{p}-\mathbf{q}} \cap \text{int}(K)$. ■

Proposition 4.2.2.1.2. *There exists an open neighbourhood of the ground point* $\mathbf{q}$ *on the boundary of* K *that is illuminated by the direction* $\mathbf{p} - \mathbf{q}$*. Denote this open neighbourhood by* $U(\mathbf{q})$.

Proof. First, the set $U(\mathbf{q})$ will be explicitly defined. In Proposition 4.2.2.1.1, it was shown that $\frac{1}{2}(\mathbf{p} + \mathbf{q}) \in \text{int}(K)$. It follows from definition that there exists a real number $\chi > 0$ such that $\text{B}(\frac{1}{2}(\mathbf{p} + \mathbf{q}), \chi) \subseteq K$. This open ball around $\frac{1}{2}(\mathbf{p} + \mathbf{q})$ will be used to generate an open 1-cylinder, which we will denote by C. Let $C = \text{B}\left(\frac{1}{2}(\mathbf{p} + \mathbf{q}), \chi\right) + \{\lambda(\mathbf{q} - \mathbf{p}) \mid \lambda \in \mathbb{R}\}$, where the set $\{\lambda(\mathbf{q} - \mathbf{p}) \mid \lambda \in \mathbb{R}\}$ is the line passing through the origin and lying parallel to the vector $\mathbf{q} - \mathbf{p}$. Now, let $U(\mathbf{q}) = C \cap \text{B}(\mathbf{q}, \chi) \cap \text{bd}(K)$.

Next, it will be verified that $U(\mathbf{q})$ contains $\mathbf{q}$ and that $U(\mathbf{q})$ is open on the bd(K). Begin by observing that the set $C \cap \text{B}(\mathbf{q}, \chi) \cap \text{bd}(K)$ can be simplified to $\text{B}(\mathbf{q}, \chi) \cap \text{bd}(K)$. This follows from the fact that $C \cap \text{B}(\mathbf{q}, \chi) = \text{B}(\mathbf{q}, \chi)$, which follows from the fact that $\text{B}(\mathbf{q}, \chi) \subseteq C$. To see that $\text{B}(\mathbf{q}, \chi) \subseteq C$, start by letting $\mathbf{x} \in \text{B}(\mathbf{q}, \chi)$ be arbitrary. This means that there exists a unit vector $\mathbf{u}$ and a scalar $0 \leqslant \eta < 1$ such that $\mathbf{x} = \mathbf{q} + \eta\chi\mathbf{u}$. The vector $\mathbf{x}$ can be re-written as $\frac{1}{2}(\mathbf{p}+\mathbf{q}) + \eta\chi\mathbf{u} + \frac{1}{2}(\mathbf{q}-\mathbf{p})$ where $\frac{1}{2}(\mathbf{p}+\mathbf{q}) + \eta\chi\mathbf{u} \in \mathrm{B}\left(\frac{1}{2}(\mathbf{p}+\mathbf{q}), \chi\right)$ and $\frac{1}{2}(\mathbf{q}-\mathbf{p}) \in \{\lambda(\mathbf{q}-\mathbf{p}) \mid \lambda \in \mathbb{R}\}$. Therefore, $\mathbf{x} \in C$.

It is now straightforward to check that $U(\mathbf{q})$ contains $\mathbf{q}$. It follows from definition that $\mathbf{q} \in \mathrm{B}(\mathbf{q}, \chi)$. From the way $\mathbf{q}$ was defined, $\mathbf{q} \in \mathrm{relbd}(B)$ and $\mathrm{relbd}(B) \subseteq \mathrm{bd}(K)$.

It follows from Theorem 1.7.1 in [60] that the open ball $\mathrm{B}(\mathbf{q}, \chi)$ is open in $\mathbb{E}^3$. Equipping $\mathrm{bd}(K)$ with the subspace topology $\mathcal{T}_{\mathrm{bd}(K)} = \{V \cap \mathrm{bd}(K) \mid V \text{ is open in } \mathbb{E}^3\}$, it can be seen that the set $U(\mathbf{q}) = \mathrm{B}(\mathbf{q}, \chi) \cap \mathrm{bd}(K) \in \mathcal{T}_{\mathrm{bd}(K)}$ and therefore, is open in $\mathrm{bd}(K)$.

Finally, it will be verified that $U(\mathbf{q})$ is illuminated by the direction $\mathbf{p} - \mathbf{q}$.

Let $\mathbf{y} \in U(\mathbf{q})$ be arbitrarily chosen. Then, there exists a unit vector $\mathbf{v}$ and a scalar $0 \leqslant \mu < 1$ such that $\mathbf{y} = \mathbf{q} + \mu\chi\mathbf{v}$. The element $\mathbf{y} + \frac{1}{2}(\mathbf{p}-\mathbf{q})$ of the ray $r^{\mathbf{y}}_{\mathbf{p}-\mathbf{q}}$ can be re-written as $\frac{1}{2}(\mathbf{p}+\mathbf{q}) + \mu\chi\mathbf{v}$. Note that $\frac{1}{2}(\mathbf{p}+\mathbf{q}) + \mu\chi\mathbf{v} \in \mathrm{B}\left(\frac{1}{2}(\mathbf{p}+\mathbf{q}), \chi\right)$ since $0 \leqslant \mu < 1$. It follows that $\mathbf{y} + \frac{1}{2}(\mathbf{p}-\mathbf{q}) \in r^{\mathbf{y}}_{\mathbf{p}-\mathbf{q}} \cap \mathrm{int}(K)$. ■

Note that for any line $\ell^{\dagger}$ in the x_1x_2-plane, the set $\mathrm{Pr}^{-1}\left(\ell^{\dagger}\right)$ is a plane in $\mathbb{E}^3$. The set $\left(W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o}, 1)\right)$ is called the *outer parallel domain* of $W_{\ell' \cap \mathrm{relbd}(B)}$ at distance $\chi' > 0$ and represents the the open neighbourhood on the $\mathrm{bd}(K)$ containing $W_{\ell' \cap \mathrm{relbd}(B)}$ that can be illuminated by the same direction or directions as $W_{\ell' \cap \mathrm{relbd}(B)}$. In this particular case,

$$\left(W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o}, 1)\right) \cap \mathrm{bd}(K) = U(\mathbf{q})$$

and it was verified in Proposition 4.2.2.1.2 that $U(\mathbf{q})$ is an open neighbourhood on $\mathrm{bd}(K)$ that can be illuminated by the direction $\mathbf{p} - \mathbf{q}$. The following lemma is proved in a general way, not using any of the assumptions particular to this case, so that can be used in all future cases; namely, the lemma holds for $0 \leqslant \dim\Big(\mathrm{aff}\left(\mathrm{Pr}^{-1}(\ell') \cap K\right)\Big) \leqslant 2$.

Lemma 4.2.2.1.3. *There exists a line ℓ_* strictly between and parallel to ℓ and ℓ' in the x_1x_2-plane such that*

$$\mathrm{slab}\left[\mathrm{Pr}^{-1}(\ell_*), \mathrm{Pr}^{-1}(\ell')\right] \cap \mathrm{bd}(K) \subseteq \left(W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o}, 1)\right) \cap \mathrm{bd}(K).$$

Proof. Since $\mathbf{q} \in \text{relbd}(B) \subseteq B = \text{cl}(B)$, it follows from Theorem 2.6.3 that there exists a sequence of points in B converging to $\mathbf{q}$. Denote this sequence by $\{\mathbf{s}_n\}_{n\in\mathbb{N}}$. For each element $\mathbf{s}_n$ of the sequence $\{\mathbf{s}_n\}_{n\in\mathbb{N}}$, let ℓ_n be the line passing through the vector $\mathbf{s}_n \in B$ which is parallel to the lines ℓ and ℓ'; this creates a sequence of parallel lines which is denoted by $\{\ell_n\}_{n\in\mathbb{N}}$. For each line ℓ_n from the sequence $\{\ell_n\}_{n\in\mathbb{N}}$, take the pre-image of the projection map onto the x_1x_2-plane of ℓ_n; this creates a sequence of hyperplanes $\{\text{Pr}^{-1}(\ell_n)\}_{n\in\mathbb{N}}$. Intersect the convex body K with each hyperplane $\text{Pr}^{-1}(\ell_n)$ from the sequence $\{\text{Pr}^{-1}(\ell_n)\}_{n\in\mathbb{N}}$; this produces the sequence $\{\text{Pr}^{-1}(\ell_n) \cap K\}_{n\in\mathbb{N}}$.

Using the Blaschke Selection Theorem, it will be shown that

$$\delta\left(\text{Pr}^{-1}(\ell_n) \cap K, \text{Pr}^{-1}(\ell') \cap K\right) \to 0$$

as $n \to \infty$. In order to be able to apply the Blaschke Selection Theorem later, one must verify that the set $\text{Pr}^{-1}(\ell') \cap K$ and every element of the sequence $\{\text{Pr}^{-1}(\ell_n) \cap K\}_{n\in\mathbb{N}}$ are compact, convex and non-empty sets.

Hyperplanes are affine and thus, convex. Of course, recall that K is also convex. These facts combined with Theorem 2.10.2 imply that $\text{Pr}^{-1}(\ell') \cap K$ and every element of the sequence $\{\text{Pr}^{-1}(\ell_n) \cap K\}_{n\in\mathbb{N}}$ are convex.

By Theorem 2.5.1, hyperplanes are closed. Thus, $\text{Pr}^{-1}(\ell')$ and every element of the sequence $\{\text{Pr}^{-1}(\ell_n)\}_{n\in\mathbb{N}}$ are closed. Recall that $K \subset \mathbb{E}^n$ is compact. Thus, K is closed. This together with *(ii)* of Theorem 2.5.1 means that $\text{Pr}^{-1}(\ell') \cap K$ and every element of the sequence $\{\text{Pr}^{-1}(\ell_n) \cap K\}_{n\in\mathbb{N}}$ are closed. Moreover, K contains $\text{Pr}-1(\ell') \cap K$ and every element of the sequence $\{\text{Pr}^{-1}(\ell_n) \cap K\}_{n\in\mathbb{N}}$. Hence, $\text{Pr}^{-1}(\ell') \cap K$ and every element of the sequence $\{\text{Pr}^{-1}(\ell_n) \cap K\}_{n\in\mathbb{N}}$ are compact by Theorem 2.8.2.

By definition, it follows that for each element $\text{Pr}^{-1}(\ell_n) \cap K$ of the sequence $\{\text{Pr}^{-1}(\ell_n) \cap K\}_{n\in\mathbb{N}}$, $\mathbf{s}_n \in \text{Pr}^{-1}(\ell_n) \cap K$ with $\mathbf{s}_n \in \{\mathbf{s}_n\}_{n\in\mathbb{N}}$ where $\{\mathbf{s}_n\}_{n\in\mathbb{N}}$ is the sequence of points belonging to B that converge to $\mathbf{q}$. This means each element of the sequence $\{\text{Pr}^{-1}(\ell_n) \cap K\}_{n\in\mathbb{N}}$ is non-empty. Also, $\{\mathbf{q}\} \in \text{Pr}^{-1}(\ell') \cap K$ and therefore, $\text{Pr}^{-1}(\ell') \cap K \neq \varnothing$.

Sub-Lemma 4.2.2.1.4. *$\delta\left(\Pr^{-1}(\ell_n)\cap K, \Pr^{-1}(\ell')\cap K\right)\to 0$ as $n\to\infty$.*

Proof. By the The Blaschke Selection Theorem, there exists a subsequence, denoted by $\{\Pr^{-1}(\ell_{n_i})\cap K\}_{i\in\mathbb{N}}$, of the sequence $\{\Pr^{-1}(\ell_n)\cap K\}_{n\in\mathbb{N}}$ that converges to $\Pr^{-1}(\ell')\cap K$. Specifically,

$$\delta\left(\Pr^{-1}(\ell_{n_i})\cap K, \Pr^{-1}(\ell')\cap K\right)\to 0 \quad \text{as } i\to\infty. \tag{4.4}$$

This means, by Theorem 3.1.6, that

(i) each point in $\Pr^{-1}(\ell')\cap K$ is the limit of a sequence $\{\mathbf{h}_{n_i}\}_{i\in\mathbb{N}}$ with $\mathbf{h}_{n_i}\in \Pr^{-1}(\ell_{n_i})\cap K$, and

(ii) the limit of any convergent sequence $\{\mathbf{h}_{n_{i_j}}\}_{j\in\mathbb{N}}$ with $\mathbf{h}_{n_{i_j}}\in \Pr^{-1}(\ell_{n_{i_j}})\cap K$ belongs to $\Pr^{-1}(\ell')\cap K$.

Let $\mathbf{h}\in \Pr^{-1}(\ell')\cap K$ be arbitrarily chosen. It follows from (i) above that there exists a sequence $\{\mathbf{h}_{n_i}\}_{i\in\mathbb{N}}$ with $\mathbf{h}_{n_i}\in \Pr^{-1}(\ell_{n_i})\cap K$ that converges to $\mathbf{h}$.

A sequence, which contains the subsequence $\{\mathbf{h}_{n_i}\}_{i\in\mathbb{N}}$, will be created as follows. If $n_1>1$, then add the points $\mathbf{h}_m$ to the sequence $\{\mathbf{h}_{n_i}\}_{n\in\mathbb{N}}$ so that the indices of this new sequence form a strictly increasing sequence of positive integers and which satisfy $\{\mathbf{h}_m\} = [\mathbf{p},\mathbf{h}_{n_1})\cap\left(\Pr^{-1}(\ell_m)\cap K\right)$ with $\Pr^{-1}(\ell_m)\cap K\in\{\Pr^{-1}(\ell_n)\cap K\}_{n\in\mathbb{N}}$, for all $1\leqslant m<n_1$. For all $n_i<m<n_{i+1}$, add the points $\mathbf{h}_m$ to the sequence $\{\mathbf{h}_{n_i}\}_{n\in\mathbb{N}}$ so that the indices of this new sequence form a strictly increasing sequence of positive integers, where $\{\mathbf{h}_m\} = (\mathbf{h}_{n_i},\mathbf{h}_{n_{i+1}})\cap\left(\Pr^{-1}(\ell_m)\cap K\right)$ with $\Pr^{-1}(\ell_m)\cap K\in\{\Pr^{-1}(\ell_n)\cap K\}_{n\in\mathbb{N}}$.

From each element $\Pr^{-1}(\ell_m)\cap K$ of $\{\Pr^{-1}(\ell_n)\cap K\}_{n\in\mathbb{N}}$ there is a corresponding element of this new sequence: either $\mathbf{h}_m$ comes from the subsequence $\{\mathbf{h}_{n_i}\}_{i\in\mathbb{N}}$, if there exists $i\in\mathbb{N}$ such that $m=n_i$; or $\mathbf{h}_m=(1-\varkappa)\mathbf{p}+\varkappa\mathbf{h}_{n_1}$ for some $0\leqslant\varkappa<1$, if $1\leqslant m<n_1$; or $\mathbf{h}_m=(1-\varkappa)\mathbf{h}_{n_i}+\varkappa\mathbf{h}_{n_k}$ for some $0<\varkappa<1$, if $n_i<m<n_k$, given that n_k is the smallest positive integer satisfying $n_i<m<n_k$ such that there exists at least one element $\mathbf{h}_m\in(\mathbf{h}_{n_i},\mathbf{h}_{n_k})$ from $\Pr^{-1}(\ell_m)\cap K$, which belongs to $\{\Pr^{-1}(\ell_n)\cap K\}_{n\in\mathbb{N}}$ but does not belong to $\{\Pr^{-1}(\ell_{n_i})\cap K\}_{i\in\mathbb{N}}$. Therefore, the new sequence may be denoted by $\{\mathbf{h}_n\}_{n\in\mathbb{N}}$.

Due the fact that $\{\mathbf{h}_{n_i}\}_{i\in\mathbb{N}}$ converges to $\mathbf{h}$, it follows that for all $\varepsilon' > 0$ there exists a real number N' such that for all $n_i > N'$,

$$\|\mathbf{h}_{n_i} - \mathbf{h}\| < \frac{\varepsilon'}{1+2\varkappa}, \tag{4.5}$$

where $0 < \varkappa < 1$ is chosen such that $\mathbf{h}_m = (1-\varkappa)\mathbf{h}_{n_i} + \varkappa\mathbf{h}_{n_k}$, given that n_k is the smallest positive integer satisfying $n_i < m < n_k$ such that there exists at least one element $\mathbf{h}_m \in (\mathbf{h}_{n_i}, \mathbf{h}_{n_k})$ from $\mathrm{Pr}^{-1}(\ell_m) \cap K$, which belongs to $\{\mathrm{Pr}^{-1}(\ell_n) \cap K\}_{n\in\mathbb{N}}$ but does not belong to $\{\mathrm{Pr}^{-1}(\ell_{n_i}) \cap K\}_{i\in\mathbb{N}}$. Observe that

$$\begin{aligned}
\|\mathbf{h}_m - \mathbf{h}\| &= \|\mathbf{h}_{n_i} - \mathbf{h} + \mathbf{h}_m - \mathbf{h}_{n_i}\| \\
&\leqslant \|\mathbf{h}_{n_i} - \mathbf{h}\| + \|\mathbf{h}_m - \mathbf{h}_{n_i}\| \\
&\leqslant \|\mathbf{h}_{n_i} - \mathbf{h}\| + \varkappa\|\mathbf{h}_{n_k} - \mathbf{h}_{n_i}\| \\
&\leqslant \|\mathbf{h}_{n_i} - \mathbf{h}\| + \varkappa\left(\|\mathbf{h} - \mathbf{h}_{n_i}\| + \|\mathbf{h}_{n_k} - \mathbf{h}\|\right) \\
&< \frac{\varepsilon'}{1+2\varkappa} + \frac{2\varkappa\varepsilon'}{1+2\varkappa} = \varepsilon'.
\end{aligned}$$

This implies that the sequence $\{\mathbf{h}_n\}_{n\in\mathbb{N}}$ converges to $\mathbf{h}$. Hence, each point in $\mathrm{Pr}^{-1}(\ell') \cap K$ is the limit of a sequence $\{\mathbf{h}_n\}_{n\in\mathbb{N}}$ with $\mathbf{h}_n \in \mathrm{Pr}^{-1}(\ell_n) \cap K$.

Let $\{\mathbf{h}_{n_m}\}_{m\in\mathbb{N}}$ be an arbitrarily chosen convergent sequence with $\mathbf{h}_{n_m} \in \mathrm{Pr}^{-1}(\ell_{n_m}) \cap K$. Suppose for a contradiction that $\{\mathbf{h}_{n_m}\}_{m\in\mathbb{N}}$ converges to $\mathbf{h}' \notin \mathrm{Pr}^{-1}(\ell') \cap K$.

A new sequence, which contains the subsequence $\{\mathbf{h}_{n_{i_j}}\}_{j\in\mathbb{N}}$ with $\mathbf{h}_{n_{i_j}} \in \mathrm{Pr}^{-1}(\ell_{n_{i_j}}) \cap K$, will be created as follows.

For all $n_m < n_i < n_{m+1}$, add the points $\mathbf{h}_{n_i}$ to the sequence $\{\mathbf{h}_{n_m}\}_{m\in\mathbb{N}}$ so that the indices of the new sequence form a strictly increasing sequence of positive integers, where

$$\{\mathbf{h}_{n_i}\} = (\mathbf{h}_{n_m}, \mathbf{h}_{n_{m+1}}) \cap \left(\mathrm{Pr}^{-1}(\ell_{n_i}) \cap K\right)$$

and $\mathrm{Pr}^{-1}(\ell_{n_i}) \cap K$ is a member of the convergent subsequence $\{\mathrm{Pr}^{-1}(\ell_{n_i}) \cap K\}_{i\in\mathbb{N}}$.

Due to the supposition that $\{\mathbf{h}_{n_m}\}_{m\in\mathbb{N}}$ converges to $\mathbf{h}' \notin \mathrm{Pr}^{-1}(\ell') \cap K$, it follows that for all $\epsilon > 0$, there exists a real number $\widetilde{N}$ such that for all $n_m > \widetilde{N}$,

$$\|\mathbf{h}_{n_m} - \mathbf{h}'\| < \frac{\epsilon}{1+2\varkappa},$$

where $0 < \varkappa < 1$ is chosen such that $\mathbf{h}_{n_i} = (1-\varkappa)\mathbf{h}_{n_m} - \varkappa\mathbf{h}_{n_k}$, given that n_k is the smallest positive integer satisfying $n_m < n_i < n_k$ such that there exists at least one element $\mathbf{h}_{n_i} \in (\mathbf{h}_{n_m}, \mathbf{h}_{n_k})$ from $\mathrm{Pr}^{-1}(\ell_{n_i}) \cap K$. Observe that

$$\begin{aligned}
\|\mathbf{h}_{n_i} - \mathbf{h}'\| &= \|\mathbf{h}_{n_m} - \mathbf{h}' + \mathbf{h}_{n_i} - \mathbf{h}_{n_m}\| \\
&\leqslant \|\mathbf{h}_{n_m} - \mathbf{h}'\| + \|\mathbf{h}_{n_i} - \mathbf{h}_{n_m}\| \\
&= \|\mathbf{h}_{n_m} - \mathbf{h}'\| + \varkappa\|\mathbf{h}_{n_k} - \mathbf{h}_{n_m}\| \\
&\leqslant \|\mathbf{h}_{n_m} - \mathbf{h}'\| + \varkappa\left(\|\mathbf{h}_{n_k} - \mathbf{h}'\| + \|\mathbf{h}' - \mathbf{h}_{n_m}\right) \\
&< \frac{\epsilon}{1+2\varkappa} + \frac{2\varkappa\epsilon}{1+2\varkappa} = \epsilon.
\end{aligned}$$

This implies that the newly created sequence converges to $\mathbf{h}'$. The newly created sequence contains the subsequence $\{\mathbf{h}_{n_{i_j}}\}_{j\in\mathbb{N}}$ with $\mathbf{h}_{n_{i_j}} \in \mathrm{Pr}^{-1}(\ell_{n_{i_j}}) \cap K$. This coupled with Theorem 2.6.1 implies that $\{\mathbf{h}_{n_{i_j}}\}_{j\in\mathbb{N}}$ converges to $\mathbf{h}' \notin \mathrm{Pr}^{-1}(\ell') \cap K$. However, this contradicts (ii). Hence, the limit of any convergent sequence $\{\mathbf{h}_{n_m}\}_{m\in\mathbb{N}}$ with $\mathbf{h}_{n_m} \in \mathrm{Pr}^{-1}(\ell_{n_m}) \cap K$ belongs to $\mathrm{Pr}^{-1}(\ell') \cap K$.

Therefore, it follows from Theorem 3.1.6 that $\delta\left(\mathrm{Pr}^{-1}(\ell_n) \cap K, \mathrm{Pr}^{-1}(\ell') \cap K\right) \to 0$ as $n \to \infty$. ■

Sub-Lemma 4.2.2.1.5. *The closed convex curve* $\mathrm{Pr}^{-1}(\ell_n) \cap \mathrm{bd}(K)$ *coincides with the set* $\mathrm{relbd}\left(\mathrm{Pr}^{-1}(\ell_n) \cap K\right)$ *in the plane* $\mathrm{Pr}^{-1}(\ell_n)$.

Proof. Let $\mathbf{x} \in \mathrm{relbd}\left(\mathrm{Pr}^{-1}(\ell_n) \cap K\right)$ be arbitrarily chosen. By definition,

$$\mathrm{relbd}\left(\mathrm{Pr}^{-1}(\ell_n) \cap K\right) = \mathrm{cl}\left(\mathrm{Pr}^{-1}(\ell_n) \cap K\right) \backslash \mathrm{relint}\left(\mathrm{Pr}^{-1}(\ell_n) \cap K\right).$$

Then,

$$\mathbf{x} \in \mathrm{cl}\left(\mathrm{Pr}^{-1}(\ell_n) \cap K\right) \qquad \text{and} \qquad \mathbf{x} \notin \mathrm{relint}\left(\mathrm{Pr}^{-1}(\ell_n) \cap K\right). \tag{4.6}$$

Recall from above that each $\mathrm{Pr}^{-1}(\ell_n) \cap K$ is compact, convex and non-empty. It follows from Theorem 2.5.1 and $\mathrm{Pr}^{-1}(\ell_n) \cap K$ being closed that $\mathbf{x} \in \mathrm{Pr}^{-1}(\ell_n) \cap K$. Therefore,

$$\mathbf{x} \in \mathrm{Pr}^{-1}(\ell_n) \qquad \text{and} \qquad x \in K. \tag{4.7}$$

It follows from K being closed and Theorem 2.5.7 that $K = \mathrm{int}(K) \cup \mathrm{bd}(K)$ where $\mathrm{int}(K) \cap \mathrm{bd}(K) = \varnothing$. This means that either $\mathbf{x} \in \mathrm{int}(K)$ or $\mathbf{x} \in \mathrm{bd}(K)$. Suppose for a contradiction that $\mathbf{x} \in \mathrm{int}(K)$. Then there exists a real number $r > 0$ such that $\mathrm{B}(\mathbf{x}, r) \subseteq K$. It follows that

$$\mathrm{B}(\mathbf{x}, r) \cap \mathrm{Pr}^{-1}(\ell_n) \subseteq K \cap \mathrm{Pr}^{-1}(\ell_n).$$

This means that $\mathbf{x} \in \mathrm{relint}\left(\mathrm{Pr}^{-1}(\ell_n) \cap K\right)$, which is a contradiction. Therefore, $\mathbf{x} \in \mathrm{bd}(K)$. This together with 4.7 implies that $\mathbf{x} \in \mathrm{Pr}^{-1}(\ell_n) \cap \mathrm{bd}(K)$. ■

Sub-Lemma 4.2.2.1.6. *Any arbitrary sequence of vectors* $\{\mathbf{r}_n\}_{n \in \mathbb{N}}$ *from* $\{\mathrm{relbd}(\mathrm{Pr}^{-1}(\ell_n) \cap K)\}_{n \in \mathbb{N}}$ *converges to* $\mathbf{q}$ *as* $n \to \infty$.

Proof. Since $\mathrm{relbd}(\mathrm{Pr}^{-1}(\ell_n) \cap K) \subseteq \mathrm{Pr}^{-1}(\ell_n) \cap K$, it follows that

$$\|\mathbf{r}_n - \mathbf{q}\| \leqslant \delta(\mathrm{Pr}^{-1}(\ell_n) \cap K, \mathrm{Pr}^{-1}(\ell') \cap K) = \max\{\|\mathbf{y}_n - \mathbf{q}\| \mid \mathbf{y}_n \in \mathrm{Pr}^{-1}(\ell_n) \cap K\} < \varepsilon.$$

■

Sub-Lemma 4.2.2.1.7. *There exists an* $\mathcal{N} \in \mathbb{R}$ *such that* $n \geqslant \mathcal{N}$ *implies* $\mathrm{relbd}(\mathrm{Pr}^{-1}(\ell_n) \cap K) \subseteq U(\mathbf{q})$.

Proof. Since $U(\mathbf{q})$ is an open neighbourhood around $\mathbf{q}$ in $\mathrm{bd}(K)$, it follows that

$$\mathrm{relbd}(U(\mathbf{q})) \nsubseteq U(\mathbf{q}).$$

Thus,

$$\mathrm{d}\big(\mathrm{relbd}(U(\mathbf{q})), \mathbf{q}\big) = \inf\{\|\mathbf{u} - \mathbf{q}\| \mid \mathbf{u} \in \mathrm{relbd}(U(\mathbf{q}))\} > 0.$$

Furthermore, $\mathrm{relbd}(U(\mathbf{q})) \subseteq \mathrm{bd}(K)$. To see this, first observe that $\mathrm{bd}(K)$ is closed since it can be written as the intersection of two closed sets by definition. Also, $\mathrm{cl}(U(\mathbf{q}))$ is the intersection of all closed sets containing the set $U(\mathbf{q})$. Thus, $\mathrm{cl}(U(\mathbf{q})) \subseteq \mathrm{bd}(K)$. Finally, note that $\mathrm{relbd}(U(\mathbf{q})) \subseteq \mathrm{cl}(U(\mathbf{q}))$, since $\mathrm{cl}(U(\mathbf{q})) = \mathrm{relint}(U(\mathbf{q})) \cup \mathrm{relbd}(U(\mathbf{q}))$.

By Claim 4, there exists an $\mathcal{N} \in \mathbb{R}$ such that $n \geqslant \mathcal{N}$ implies $\|\mathbf{r}_n - \mathbf{q}\| < \mathrm{d}(\mathrm{relbd}(U(\mathbf{q})), \mathbf{q})$, for any $\mathbf{r}_n \in \mathrm{relbd}(\mathrm{Pr}^{-1}(\ell_n) \cap K)$. This means that $\mathrm{relbd}(\mathrm{Pr}^{-1}(\ell_n) \cap K) \subseteq U(\mathbf{q})$, for $n \geqslant \mathcal{N}$. ■

In other words, $\mathrm{Pr}^{-1}(\ell_n) \cap \mathrm{bd}(K) \subseteq U(\mathbf{q})$, for $n \geqslant \mathcal{N}$. It follows directly that

$$\mathrm{slab}\left[\mathrm{Pr}^{-1}(\ell_{\mathcal{N}}), \mathrm{Pr}^{-1}(\ell')\right] \cap \mathrm{bd}(K) \subseteq U(\mathbf{q}), \text{ for } n \geqslant \mathcal{N}.$$

Let $\ell_* = \ell_{\mathcal{N}}$. The line ℓ_* lies in the x_1x_2-plane strictly between and parallel to ℓ and ℓ' such that $\mathrm{slab}\left[\mathrm{Pr}^{-1}(\ell_*), \mathrm{Pr}^{-1}(\ell')\right] \cap \mathrm{bd}(K) \subseteq U(\mathbf{q})$. ■

Proposition 4.2.2.1.8. *The line ℓ_* intersects* relbd(B) *at exactly two points.*

Proof. Recall that the line ℓ' is parallel to but distinct from the line ℓ. This means there exists a vector $\mathbf{t} \neq \mathbf{o}$ such that $\ell' = \ell + \mathbf{t}$. Due to the fact that ℓ_* is parallel to and lies strictly between ℓ and ℓ', there exists $0 < \gamma < 1$ such that $\ell_* = \ell' - \gamma\mathbf{t} = (1-\gamma)\ell' + \gamma\ell$. Recall that $\mathbf{p} \in \ell \cap B$ and $\mathbf{q} \in \ell' \cap B$. Thus, $\gamma\mathbf{p} + (1-\gamma)\,\mathbf{q} \in \ell_* \cap (\mathbf{p}, \mathbf{q})$. Moreover, recall that $(\mathbf{p}, \mathbf{q}) \subseteq \mathrm{relint}(B)$. For the sake of simplicity, denote $\gamma\mathbf{p} + (1-\gamma)\,\mathbf{q}$ by $\mathbf{y}$. Thus, $\mathbf{y} \in \ell_* \cap \mathrm{relint}(B)$.
Note that the line ℓ_* can be written as the union of two rays emanating from $\mathbf{y} \in \ell_* \cap \mathrm{relint}(B)$ with opposite directions. By 2.32 in Appendix 1 of [3], each of these rays emanating from $\mathbf{y}$ will intersect the relative boundary of B at exactly one point. Thus, ℓ_* will intersect the relative boundary of B at two distinct points. ■

One of the points of the set $\ell_* \cap \mathrm{relbd}(B)$ lies in $(\mathbf{p}, \mathbf{q})_B$, denote it by $\mathbf{a}$, and the other lies in $(\mathbf{q}, \mathbf{p})_B$, denote it by $\mathbf{b}$. In particular, this means that while travelling counter-clockwise on the simple closed curve relbd(B) from the starting point $\mathbf{p}$, the points $\mathbf{a}$, $\mathbf{q}$, $\mathbf{b}$ follow each other in this order, before one returns to $\mathbf{p}$.

Proposition 4.2.2.1.9. *The open line segment $(\mathbf{a}, \mathbf{b})$ is contained by $\ell_* \cap \mathrm{relint}(B)$.*

Proof. Recall that ℓ_* can be written as the union of two rays emanating from $\mathbf{y} \in \ell_* \cap \mathrm{relint}(B)$. Note that the half-open line segment $(\mathbf{a}, \mathbf{y}]$ belongs to one of these two rays and $[\mathbf{y}, \mathbf{b})$ belongs to the other. It follows from Theorem 2.10.10 that $(\mathbf{a}, \mathbf{y}], [\mathbf{y}, \mathbf{b}) \subseteq \mathrm{relint}(B)$. Therefore, $(\mathbf{a}, \mathbf{b}) = (\mathbf{a}, \mathbf{y}] \cup [\mathbf{y}, \mathbf{b}) \subseteq \mathrm{relint}(B)$. ■

As a consequence of Proposition 4.2.2.1.9, there exists a real number $0 < \xi < 1$ such that $\mathbf{y} = \xi \mathbf{a} + (1-\xi)\,\mathbf{b}$.

Let $B_{\mathbf{p}}$ and $B_{\mathbf{q}}$ denote the compact parts of B separated by the line segment $[\mathbf{a}, \mathbf{b}]$ such that $\mathbf{p} \in B_{\mathbf{p}}$, $\mathbf{q} \in B_{\mathbf{q}}$, $B_{\mathbf{p}} \cap B_{\mathbf{q}} = [\mathbf{a}, \mathbf{b}]$ and $B_{\mathbf{p}} \cup B_{\mathbf{q}} = B$. More formally, $B_{\mathbf{q}} = B \cap \operatorname{slab}\left[\operatorname{Pr}^{-1}(\ell_*), \operatorname{Pr}^{-1}(\ell')\right]$ and $B_{\mathbf{p}} = B \cap \operatorname{slab}\left[\operatorname{Pr}^{-1}(\ell), \operatorname{Pr}^{-1}(\ell_*)\right]$. Note that $B_{\mathbf{q}} \cap \operatorname{relbd}(B) = [\mathbf{a}, \mathbf{b}]_B$ and $B_{\mathbf{p}} \cap \operatorname{relbd}(B) = [\mathbf{b}, \mathbf{a}]_B$.

Furthermore, let $B_{\mathbf{a}}$ and $B_{\mathbf{b}}$ denote the compact subsets of B separated by the line segment $[\mathbf{p}, \mathbf{q}]$ such that $\mathbf{a} \in B_{\mathbf{a}}$, $\mathbf{b} \in B_{\mathbf{b}}$, $B_{\mathbf{a}} \cap B_{\mathbf{b}} = [\mathbf{p}, \mathbf{q}]$ and $B_{\mathbf{a}} \cup B_{\mathbf{b}} = B$. Specifically, $B_{\mathbf{a}} = B \cap \{(1-\Gamma)\,\mathbf{p} + \Gamma \mathbf{q} + \lambda\,(\mathbf{a} - \mathbf{b}) \mid 0 \leqslant \Gamma \leqslant 1,\ \lambda \geqslant 0\}$ and $B_{\mathbf{b}} = B \cap \{(1-\Gamma)\,\mathbf{p} + \Gamma \mathbf{q} + \lambda'\,(\mathbf{b} - \mathbf{a}) \mid 0 \leqslant \Gamma \leqslant 1,\ \lambda' \geqslant 0\}$. Note that $B_{\mathbf{a}} \cap \operatorname{relbd}(B) = [\mathbf{p}, \mathbf{q}]_B$ and $B_b \cap \operatorname{relbd}(B) = [\mathbf{q}, \mathbf{p}]_B$.

Proposition 4.2.2.1.10. *All elements in* $B_{\mathbf{q}} \cap \operatorname{relbd}(B)$ *are illuminated by the direction* $\mathbf{p} - \mathbf{q}$.

Proof. Note that $B_{\mathbf{q}} \subseteq \operatorname{slab}\left[\operatorname{Pr}^{-1}(\ell_*), \operatorname{Pr}^{-1}(\ell')\right]$. Recall that $\operatorname{relbd}(B) \subseteq \operatorname{bd}(K)$. Hence, $B_{\mathbf{q}} \cap \operatorname{relbd}(B) \subseteq \operatorname{slab}\left[\operatorname{Pr}^{-1}(\ell_*), \operatorname{Pr}^{-1}(\ell')\right] \cap \operatorname{bd}(K)$. It follows from Lemma 4.2.2.1.3 that $B_{\mathbf{q}} \cap \operatorname{relbd}(B) \subseteq U(\mathbf{q})$. This together with Proposition 4.2.2.1.2 implies that $B_{\mathbf{q}} \cap \operatorname{relbd}(B)$ is illuminated by the direction $\mathbf{p} - \mathbf{q}$. ■

Proposition 4.2.2.1.11. *It is useful to note that* $\operatorname{relbd}(B) = \big(B_{\mathbf{p}} \cap \operatorname{relbd}(B)\big) \cup \big(B_{\mathbf{q}} \cap \operatorname{relbd}(B)\big)$.

Proof. Due to the basic fact that set intersection is distributive over set union, it follows that

$$\big(B_{\mathbf{p}} \cap \operatorname{relbd}(B)\big) \cup \big(B_{\mathbf{q}} \cap \operatorname{relbd}(B)\big) = \operatorname{relbd}(B) \cap \big(B_{\mathbf{p}} \cup B_{\mathbf{q}}\big).$$

Substitute B for $B_{\mathbf{p}} \cup B_{\mathbf{q}}$, to get

$$\operatorname{relbd}(B) \cap (B_{\mathbf{p}} \cup B_{\mathbf{q}}) = \operatorname{relbd}(B) \cap B.$$

Since $\operatorname{relbd}(B) \subseteq B$, it follows that $\operatorname{relbd}(B) \cap B = \operatorname{relbd}(B)$. ■

Notice that the line ℓ can be written as $\{\mathbf{p} + \lambda(\mathbf{a} - \mathbf{b}) \mid \lambda \in \mathbb{R}\}$ or equivalently as $\{\mathbf{p} + (-\lambda)(\mathbf{b} - \mathbf{a}) \mid (-\lambda) \in \mathbb{R}\}$.

Let $\mathcal{T} = \max\{\|\mathbf{k}^+ - \mathbf{k}^-\| \mid \mathbf{k} \in K\}$, where $\mathbf{k}^+$ is the endpoint of the non-degenerate line segment $\text{Pr}(\mathbf{k}) \cap K$ lying in $H_{\mathbf{+}}$ and $\mathbf{k}^-$ is the other endpoint of that line segment lying in $H_{\mathbf{-}}$.

Lemma 4.2.2.1.12. *The following six directions will illuminate* $\text{bd}(K)\backslash U(\mathbf{q})$*:* $\frac{1}{2}(\mathbf{q}+\mathbf{a})-\mathbf{b}$*,* $\frac{1}{2}(\mathbf{q}+\mathbf{b})-\mathbf{a}$*,* $\left(1-\frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{a})-\mathbf{b}\right) \pm \mathcal{T}\mathbf{e}_3$ *and* $\left(1-\frac{2(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{b})-\mathbf{a}\right) \pm \mathcal{T}\mathbf{e}_3$*.*

Proof. Let $\mathbf{x} \in \text{bd}(K)\backslash U(\mathbf{q})$ be arbitrarily chosen. This means $\mathbf{x} \in \text{bd}(K)$ but $\mathbf{x} \notin U(\mathbf{q})$. Recall from Proposition 4.1.2 that $\text{bd}(K) = W \cup \text{relint}(B)_{\mathbf{+}} \cup \text{relint}(B)_{\mathbf{-}}$ and that W, $\text{relint}(B)_{\mathbf{+}}$ and $\text{relint}(B)_{\mathbf{-}}$ are pairwise disjoint. Thus, either $\mathbf{x} \in W$ or $\mathbf{x} \in \text{relint}(B)_{\mathbf{+}}$ or $\mathbf{x} \in \text{relint}(B)_{\mathbf{-}}$. It follows from $\mathbf{x} \notin U(\mathbf{q})$ that $\mathbf{x} \notin \text{slab}\left[\text{Pr}^{-1}(\ell_*), \text{Pr}^{-1}(\ell')\right] \cap \text{bd}(K) \subseteq U(\mathbf{q})$. Use de Morgan's Law to get that $\mathbf{x} \notin \text{slab}\left[\text{Pr}^{-1}(\ell_*), \text{Pr}^{-1}(\ell')\right]$ or $\mathbf{x} \notin \text{bd}(K)$. However, $\mathbf{x} \in \text{bd}(K)$ and therefore, $\mathbf{x} \notin \text{slab}\left[\text{Pr}^{-1}(\ell_*), \text{Pr}^{-1}(\ell')\right]$.

<u>Case 1:</u> Suppose $\mathbf{x} \in W = \left(\text{Pr}^{-1}(\text{relbd}(B)) \cap \text{bd}(K)\right)$. Specifically, suppose $x \in \text{relbd}(B)$. In summary, $\mathbf{x} \in \text{relbd}(B) \subseteq B$ and $\mathbf{x} \notin \text{slab}\left[\text{Pr}^{-1}(\ell_*), \text{Pr}^{-1}(\ell')\right]$. Recall that $B_{\mathbf{q}} \subseteq \text{slab}\left[\text{Pr}^{-1}(\ell_*), \text{Pr}^{-1}(\ell')\right]$. It follows that $\mathbf{x} \notin B_{\mathbf{q}}$. Thus, $\mathbf{x} \in B\backslash B_{\mathbf{q}} \subseteq B_{\mathbf{p}}$ and in particular, $\mathbf{x} \in B_{\mathbf{p}} \cap \text{relbd}(B)$.

Remark. This case looks after illuminating any ground points on the wall through $B_{\mathbf{p}} \cap \text{relbd}(B)$.

The same method used to show that Proposition 4.2.2.1.11 held, can be used to show

$$B_{\mathbf{p}} \cap \text{relbd}(B) = (B_{\mathbf{a}} \cap B_{\mathbf{p}} \cap \text{relbd}(B)) \cup (B_{\mathbf{b}} \cap B_{\mathbf{p}} \cap \text{relbd}(B)).$$

Suppose, furthermore, that $\mathbf{x} \in B_{\mathbf{b}} \cap B_{\mathbf{p}} \cap \text{relbd}(B) = [\mathbf{b}, \mathbf{p}]_B$.

An argument will be presented below to show that any vector strictly between $\mathbf{a} - \mathbf{b}$ and $\mathbf{q} - \mathbf{b}$ illuminates the closed arc $B_{\mathbf{b}} \cap B_{\mathbf{p}} \cap \text{relbd}(B) = [\mathbf{b}, \mathbf{p}]_B$ of the curve $\text{relbd}(B)$ in the x_1x_2-plane.

A precise definition of a vector strictly between $\mathbf{a} - \mathbf{b}$ and $\mathbf{q} - \mathbf{b}$ is required. Denote the angle between the vectors $\mathbf{a} - \mathbf{b}$ and $\mathbf{q} - \mathbf{b}$, with value ranging from 0 to π, by α. Likewise, denote the angle between the vectors $\mathbf{q} - \mathbf{a}$ and $\mathbf{b} - \mathbf{a}$, whose value is between 0 and π, by β. It should be noted that the vectors $\mathbf{a} - \mathbf{b}$ and $\mathbf{q} - \mathbf{b}$ are not parallel. To see this recall that $\mathbf{q} \in \ell' = \ell + \mathbf{t}$ and $\mathbf{b} \in \ell_* = \ell + \gamma \mathbf{t}$ for $\mathbf{t} \neq \mathbf{o}$ and $0 < \gamma < 1$. Also, note that $\mathbf{q} - \mathbf{b} = \mathbf{p} + \lambda_1 (\mathbf{a} - \mathbf{b}) + \mathbf{t} - \mathbf{p} - \lambda_2 (\mathbf{a} - \mathbf{b}) - \gamma \mathbf{t} = (\lambda_1 - \lambda_2)(\mathbf{a} - \mathbf{b}) + (1 - \gamma)\mathbf{t}$, for some $\lambda_1, \lambda_2 \in \mathbb{R}$. It can be similarly shown that the vectors $\mathbf{q} - \mathbf{a}$ and $\mathbf{b} - \mathbf{a}$ are not parallel. This means that the angles α and β are strictly between 0 and π. Let $R_{x_3}(\alpha)$ denote the linear transformation which rotates the x_1x_2-plane counter-clockwise through an angle of α. The vector $\mathbf{q} - \mathbf{b}$ has the same direction as the vector $\mathbf{a} - \mathbf{b}$ rotated counter-clockwise through an angle of α in the x_1x_2-plane. Specifically,

$$\mathbf{q} - \mathbf{b} = \frac{\|\mathbf{q} - \mathbf{b}\|}{\|\mathbf{a} - \mathbf{b}\|} \cdot R_{x_3}(\alpha)(\mathbf{a} - \mathbf{b}) = \frac{\|\mathbf{q} - \mathbf{b}\|}{\|\mathbf{a} - \mathbf{b}\|} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} (\mathbf{a} - \mathbf{b}).$$

It should be clear that $\frac{\|\mathbf{q}-\mathbf{b}\|}{\|\mathbf{a}-\mathbf{b}\|} > 0$. A vector is strictly between the vectors $\mathbf{a} - \mathbf{b}$ and $\mathbf{q} - \mathbf{b}$ if it has the same direction as the vector $\mathbf{a} - \mathbf{b}$ rotated counter-clockwise in the x_1x_2-plane through an angle strictly less than α but strictly greater than 0.

Sub-Lemma 4.2.2.1.13. *Any vector strictly between* $\mathbf{a}-\mathbf{b}$ *and* $\mathbf{q}-\mathbf{b}$ *has the same direction as*

$$(1 - \partial)(\mathbf{a} - \mathbf{b}) + \partial(\mathbf{q} - \mathbf{b}), \text{ for some } 0 < \partial < 1.$$

Proof. First, notice that the angle between the vectors $(1 - \partial)(\mathbf{a} - \mathbf{b}) + \partial(\mathbf{q} - \mathbf{b})$ and $\mathbf{a} - \mathbf{b}$, which will be denoted by θ, is less than α. The triangle inequality is used below to get

$$\begin{aligned} \cos\theta &= \frac{\langle (1 - \partial)(\mathbf{a} - \mathbf{b}) + \partial(\mathbf{q} - \mathbf{b}), \mathbf{a} - \mathbf{b} \rangle}{\|\|\mathbf{a} - \mathbf{b}\| \cdot \| (1 - \partial)(\mathbf{a} - \mathbf{b}) + \partial(\mathbf{q} - \mathbf{b})} \\ &= \frac{(1 - \partial)\|\mathbf{a} - \mathbf{b}\|^2 + \partial \langle \mathbf{q} - \mathbf{b}, \mathbf{a} - \mathbf{b} \rangle}{\|\mathbf{a} - \mathbf{b}\| \cdot \| (1 - \partial)(\mathbf{a} - \mathbf{b}) + \partial(\mathbf{q} - \mathbf{b}) \|} \\ &> \frac{(1 - \partial)\|\mathbf{a} - \mathbf{b}\|^2 + \partial \langle \mathbf{q} - \mathbf{b}, \mathbf{a} - \mathbf{b} \rangle}{(1 - \partial)\|\mathbf{a} - \mathbf{b}\|^2 + \partial \|\mathbf{q} - \mathbf{b}\|\|\mathbf{a} - \mathbf{b}\|}. \end{aligned}$$

It follows from the Cauchy-Schwarz inequality that $\langle \mathbf{q}-\mathbf{b}, \mathbf{a}-\mathbf{b}\rangle < \|\mathbf{q}-\mathbf{b}\|\|\mathbf{a}-\mathbf{b}\|$. The inequalities are strict due to the linear independence of the vectors $\mathbf{a}-\mathbf{b}$ and $\mathbf{q}-\mathbf{b}$. Namely, recall that equality does not hold in the triangle or Cauchy-Schwarz inequalities when the vectors involved are linearly independent. This means that Proposition A.3 can be used and it implies that

$$\frac{(1-\partial)\|\mathbf{a}-\mathbf{b}\|^2 + \partial\langle \mathbf{q}-\mathbf{b}, \mathbf{a}-\mathbf{b}\rangle}{(1-\partial)\|\mathbf{a}-\mathbf{b}\|^2 + \partial\|\mathbf{q}-\mathbf{b}\|\|\mathbf{a}-\mathbf{b}\|} > \frac{\partial\langle \mathbf{q}-\mathbf{b}, \mathbf{a}-\mathbf{b}\rangle}{\partial\|\mathbf{q}-\mathbf{b}\|\|\mathbf{a}-\mathbf{b}\|} = \cos\alpha.$$

This means that $\cos\theta > \cos\alpha$.

Notice that the vectors $(1-\partial)(\mathbf{a}-\mathbf{b}) + \partial(\mathbf{q}-\mathbf{b})$ and $\mathbf{a}-\mathbf{b}$ are also linearly independent. To see this observe that

$$\begin{aligned}(1-\partial)(\mathbf{a}-\mathbf{b}) + \partial(\mathbf{q}-\mathbf{b}) &= (1-\partial)(\mathbf{a}-\mathbf{b}) + \partial((\lambda_1-\lambda_2)(\mathbf{a}-\mathbf{b}) + (1-\gamma)\mathbf{t}) \\ &= ((1-\partial) + \partial(\lambda_1-\lambda_2))(\mathbf{a}-\mathbf{b}) + \partial(1-\gamma)\mathbf{t}.\end{aligned}$$

Also, $0 < \partial(1-\gamma) < 1$ since $0 < \partial < 1$ and $0 < 1-\gamma < 1$. This, together with the Cauchy-Schwarz inequality, imply that $1 > \cos\theta$. Therefore, $1 > \cos\theta > \cos\alpha$. This means that the angle θ between $(1-\partial)(\mathbf{a}-\mathbf{b}) + \partial(\mathbf{q}-\mathbf{b})$ and $\mathbf{a}-\mathbf{b}$ lies in the interval

$$0 < \theta < \alpha < \pi.$$

Now, notice that

$$\partial = \frac{\|\partial(\mathbf{q}-\mathbf{a})\|}{\|\mathbf{q}-\mathbf{a}\|}. \tag{4.8}$$

It follows from the Law of Sines that

$$\frac{\|\mathbf{q}-\mathbf{a}\|}{\sin\alpha} = \frac{\|\mathbf{q}-\mathbf{b}\|}{\sin\beta}$$

and

$$\frac{\|\partial(\mathbf{q}-\mathbf{a})\|}{\sin\theta} = \frac{\|(1-\partial)\mathbf{a} + \partial\mathbf{q}-\mathbf{b}\|}{\sin\beta}.$$

These equations are re-arranged and substituted into 4.8 to get

$$\partial = \frac{\|(1-\partial)\mathbf{a} + \partial\mathbf{q}-\mathbf{b}\|\sin\theta}{\|\mathbf{q}-\mathbf{b}\|\sin\alpha}.$$

Let

$$A = \begin{bmatrix} 1-\partial+\partial\frac{\|\mathbf{q}-\mathbf{b}\|}{\|\mathbf{a}-\mathbf{b}\|}\cos\alpha & -\partial\frac{\|\mathbf{q}-\mathbf{b}\|}{\|\mathbf{a}-\mathbf{b}\|}\sin\alpha & 0 \\ \partial\frac{\|\mathbf{q}-\mathbf{b}\|}{\|\mathbf{a}-\mathbf{b}\|}\sin\alpha & 1-\partial+\partial\frac{\|\mathbf{q}-\mathbf{b}\|}{\|\mathbf{a}-\mathbf{b}\|}\cos\alpha & 0 \\ 0 & 0 & 1-\partial+\partial\frac{\|\mathbf{q}-\mathbf{b}\|}{\|\mathbf{a}-\mathbf{b}\|} \end{bmatrix}.$$

Then,

$$\begin{aligned} &\frac{\|\mathbf{a}-\mathbf{b}\|}{\|\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\|}\left(\left(1-\partial\right)\left(\mathbf{a}-\mathbf{b}\right)+\partial\left(\mathbf{q}-\mathbf{b}\right)\right) \\ &= \frac{\|\mathbf{a}-\mathbf{b}\|}{\|\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\|}\left(\left(1-\partial\right)\left(\mathbf{a}-\mathbf{b}\right)+\partial\frac{\|\mathbf{q}-\mathbf{b}\|}{\|\mathbf{a}-\mathbf{b}\|}R_{x_3}\left(\alpha\right)\right)\left(\mathbf{a}-\mathbf{b}\right) \\ &= \frac{\|\mathbf{a}-\mathbf{b}\|}{\|\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\|}A\left(\mathbf{a}-\mathbf{b}\right). \end{aligned} \tag{4.9}$$

Observe that

$$\begin{aligned} &\frac{\|\mathbf{a}-\mathbf{b}\|}{\|\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\|}\cdot\partial\cdot\frac{\|\mathbf{q}-\mathbf{b}\|}{\|\mathbf{a}-\mathbf{b}\|}\sin\alpha \\ &= \frac{\|\mathbf{a}-\mathbf{b}\|}{\|\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\|}\cdot\frac{\|\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\|\sin\theta}{\|\mathbf{q}-\mathbf{b}\|\sin\alpha}\cdot\frac{\|\mathbf{q}-\mathbf{b}\|}{\|\mathbf{a}-\mathbf{b}\|}\sin\alpha \\ &= \sin\theta \end{aligned}$$

and

$$\begin{aligned} &\frac{\|\mathbf{a}-\mathbf{b}\|}{\|\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\|}\left(1-\partial+\partial\frac{\|\mathbf{q}-\mathbf{b}\|}{\|\mathbf{a}-\mathbf{b}\|}\cos\alpha\right) \\ &= \frac{\|\mathbf{a}-\mathbf{b}\|}{\|\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\|}-\frac{\partial\|\mathbf{a}-\mathbf{b}\|}{\|\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\|}+\frac{\partial\|\mathbf{q}-\mathbf{b}\|}{\|\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\|}\left(\frac{\langle\mathbf{q}-\mathbf{b},\mathbf{a}-\mathbf{b}\rangle}{\|\mathbf{q}-\mathbf{b}\|\|\mathbf{a}-\mathbf{b}\|}\right) \\ &= \frac{\left(1-\partial\right)\|\mathbf{a}-\mathbf{b}\|^2+\langle\partial\left(\mathbf{q}-\mathbf{b}\right),\mathbf{a}-\mathbf{b}\rangle}{\|\left(1-\partial\right)\left(\mathbf{a}-\mathbf{b}\right)+\partial\left(\mathbf{q}-\mathbf{b}\right)\|\cdot\|\mathbf{a}-\mathbf{b}\|} \\ &= \frac{\langle\left(1-\partial\right)\left(\mathbf{a}-\mathbf{b}\right)+\partial\left(\mathbf{q}-\mathbf{b}\right),\mathbf{a}-\mathbf{b}\rangle}{\|\left(1-\partial\right)\left(\mathbf{a}-\mathbf{b}\right)+\partial\left(\mathbf{q}-\mathbf{b}\right)\|\cdot\|\mathbf{a}-\mathbf{b}\|}=\cos\theta \end{aligned}$$

The entry in the third row and third column of the matrix in 4.9 can be re-written as follows:

$$\frac{\|\mathbf{a}-\mathbf{b}\|}{\|\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\|}\left(1-\partial+\partial\frac{\|\mathbf{q}-\mathbf{b}\|}{\|\mathbf{a}-\mathbf{b}\|}\right)=\frac{\left(1-\partial\right)\|\mathbf{a}-\mathbf{b}\|+\partial\|\mathbf{q}-\mathbf{b}\|}{\|\left(1-\partial\right)\left(\mathbf{a}-\mathbf{b}\right)+\partial\left(\mathbf{q}-\mathbf{b}\right)\|}. \tag{4.10}$$

In general, Equation 4.10 is not equal to 1. In fact, Equation 4.10 is equal to 1 if and only if there exists some real number $\lambda \geqslant 0$ such that $\partial\left(\mathbf{q}-\mathbf{b}\right)=\lambda\left(1-\partial\right)\left(\mathbf{a}-\mathbf{b}\right)$. This is does not happen here; it was shown above that the vectors $\mathbf{a}-\mathbf{b}$ and $\mathbf{q}-\mathbf{b}$ are linearly independent. However, the vector $\mathbf{a}-\mathbf{b}$ lies in the x_1x_2-plane. This implies that

$$\langle \mathbf{a}-\mathbf{b}, \mathbf{e}_3 \rangle = 0 = \left(1-\partial+\partial\frac{\|\mathbf{q}-\mathbf{b}\|}{\|\mathbf{a}-\mathbf{b}\|}\right)\langle \mathbf{a}-\mathbf{b}, \mathbf{e}_3 \rangle .$$

It follows that

$$\begin{aligned}\frac{\|\mathbf{a}-\mathbf{b}\|}{\|(1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\|}\left((1-\partial)(\mathbf{a}-\mathbf{b})+\partial(\mathbf{q}-\mathbf{b})\right) &= \begin{bmatrix}\cos\theta\langle \mathbf{a}-\mathbf{b},\mathbf{e}_1\rangle-\sin\theta\langle \mathbf{a}-\mathbf{b},\mathbf{e}_2\rangle\\ \sin\theta\langle \mathbf{a}-\mathbf{b},\mathbf{e}_1\rangle+\cos\theta\langle \mathbf{a}-\mathbf{b},\mathbf{e}_2\rangle\\ 0\end{bmatrix}\\ &= \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}(\mathbf{a}-\mathbf{b})\\ &= R_{x_3}(\theta)(\mathbf{a}-\mathbf{b}).\end{aligned}$$

Recall from Proposition 4.2.2.1.8 that $\mathbf{a},\mathbf{b}\in\ell_*$ are distinct. This means that $\mathbf{a}-\mathbf{b}\neq\mathbf{o}$ and therefore, $\|\mathbf{a}-\mathbf{b}\|>0$. Also, recall that $\mathbf{q}\in\ell'$ is distinct from the points $\mathbf{a},\mathbf{b}\in\ell_*$, since it lies on a distinct parallel line. Moreover, recall that $0<\partial<1$. It follows that $(1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}=(1-\partial)(\mathbf{a}-\mathbf{b})+\partial(\mathbf{q}-\mathbf{b})\neq\mathbf{o}$ and therefore, $\|(1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\|>0$. Hence,

$$\frac{\|\mathbf{a}-\mathbf{b}\|}{\|(1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\|}>0.$$

■

Now, notice that the line passing through the points $\mathbf{b}$ and $(1-\partial)\,\mathbf{a}+\partial\mathbf{q}$ intersects with the line ℓ_* at the point $\mathbf{b}$. It follows from Theorem 2.2.2.1 that the line through the points $\mathbf{b}$ and $(1-\partial)\,\mathbf{a}+\partial\mathbf{q}$ is not parallel to the line ℓ_*. This implies that the line through the points $\mathbf{b}$ and $(1-\partial)\,\mathbf{a}+\partial\mathbf{q}$ is also not parallel to the line ℓ, since the lines ℓ and ℓ_* are parallel. The intersection point of the line through the points $\mathbf{b}$ and $(1-\partial)\,\mathbf{a}+\partial\mathbf{q}$ with ℓ is

$$\mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big(\mathbf{b}-\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}\big)\Big).$$

To see that this point lies on the line ℓ, observe the following argument:

$$\begin{aligned}
&\mathbf{b} + \frac{1-\gamma}{\partial\gamma}\Big(\mathbf{b} - \big((1-\partial)\,\mathbf{a} + \partial\mathbf{q}\big)\Big) \\
&= \mathbf{p} + (\mathbf{b}-\mathbf{p}) - \frac{1-\gamma}{\partial\gamma}\Big((1-\partial(1-\xi))(\mathbf{a}-\mathbf{b}) + \partial\gamma(\mathbf{q}-\mathbf{p})\Big) \\
&= \mathbf{p} + \xi(\mathbf{b}-\mathbf{a}) + (1-\gamma)(\mathbf{q}-\mathbf{p}) - (1-\gamma)(\mathbf{q}-\mathbf{p}) - \frac{(1-\gamma)(1-\partial(1-\xi))}{\partial\gamma}(\mathbf{a}-\mathbf{b}) \\
&= \mathbf{p} + \left(-\frac{(1-\gamma)(1-\partial(1-\xi))}{\partial\gamma} - \xi\right)(\mathbf{a}-\mathbf{b}) \in \{\mathbf{p} + \lambda(\mathbf{a}-\mathbf{b}) \mid \lambda\in\mathbb{R}\} = \ell.
\end{aligned}$$

Sub-Lemma 4.2.2.1.14.

$$B_{\mathbf{b}} \cap B_{\mathbf{p}} \subseteq \operatorname{conv}\left\{\mathbf{p}, (1-\partial)\,\mathbf{a} + \partial\mathbf{q}, \mathbf{b} + \frac{1-\gamma}{\partial\gamma}\Big(\mathbf{b} - \big((1-\partial)\,\mathbf{a} + \partial\mathbf{q}\big)\Big)\right\}.$$

Proof. Suppose for a contradiction that this is not the case.

In particular, suppose that there exists $\mathbf{z} \in B_{\mathbf{b}} \cap B_{\mathbf{p}}$ such that

$$\mathbf{z} \notin \operatorname{conv}\left\{\mathbf{p}, (1-\partial)\,\mathbf{a} + \partial\mathbf{q}, \mathbf{b} + \frac{1-\gamma}{\partial\gamma}\Big(\mathbf{b} - \big((1-\partial)\,\mathbf{a} + \partial\mathbf{q}\big)\Big)\right\}.$$

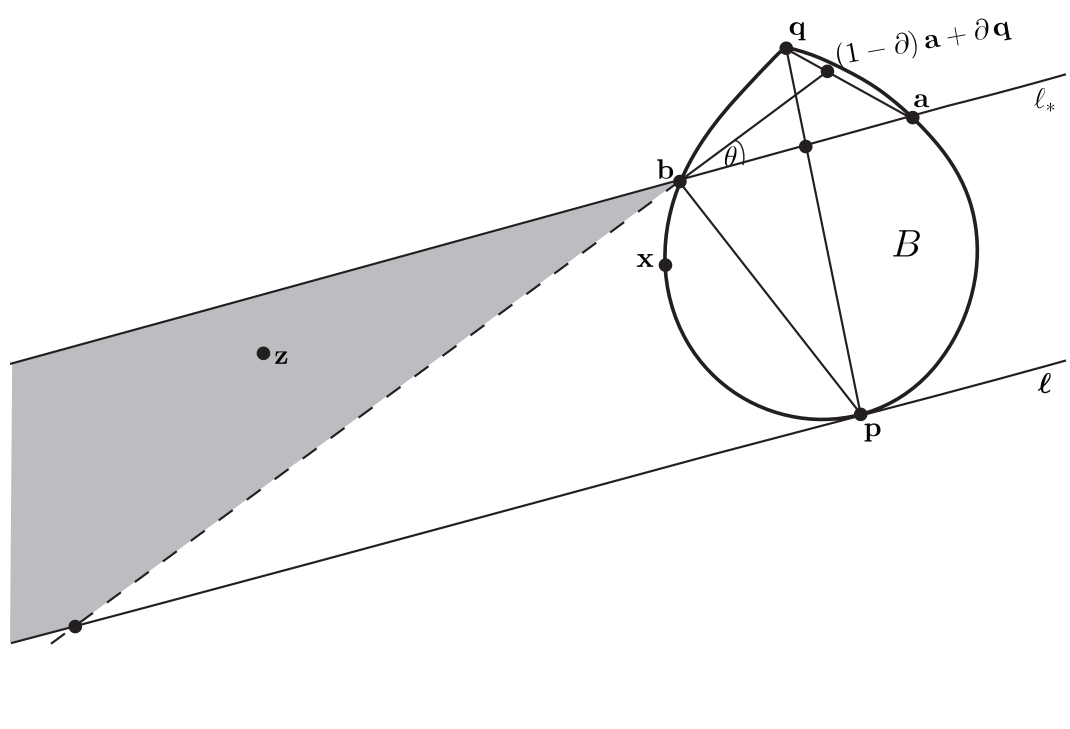


Figure 4.11: The point $\mathbf{z}$ lies somewhere in the shaded region

This means that $\mathbf{z} \in B$ and can be written as $\mathbf{p} + \mu_1 (\mathbf{q} - \mathbf{p}) + \mu_2 (\mathbf{b} - \mathbf{a})$, for some

$$0 \leqslant \mu_1 \leqslant 1 - \gamma \tag{4.11}$$

and for some

$$\mu_2 > \left(1 - \frac{\mu_1}{1-\gamma}\right)\left(\frac{(1-\gamma)(1-\partial(1-\xi))}{\partial\gamma}\right) + \xi. \tag{4.12}$$

Moreover, $\mathbf{z}$ lies on the ray emanating from the point

$$\left(\gamma - \frac{\xi(1-\gamma-\mu_1)}{\mu_2-\xi}\right)\mathbf{p} + \left(1 - \left(\gamma - \frac{\xi(1-\gamma-\mu_1)}{\mu_2-\xi}\right)\right)\mathbf{q} \in [\mathbf{y}, \mathbf{q}) \subseteq (\mathbf{p}, \mathbf{q}) \subseteq \mathrm{relint}(B)$$

with direction $\mathbf{z} - \mathbf{b}$. Specifically,

$$\begin{aligned}
&\left(\gamma - \frac{\xi(1-\gamma-\mu_1)}{\mu_2-\xi}\right)\mathbf{p} + \left(1 - \left(\gamma - \frac{\xi(1-\gamma-\mu_1)}{\mu_2-\xi}\right)\right)\mathbf{q} + \left(1 + \frac{\xi}{\mu_2-\xi}\right)(\mathbf{z} - \mathbf{b}) \\
&= \mathbf{p} + (1-\gamma)(\mathbf{q}-\mathbf{p}) + \frac{\xi(1-\gamma-\mu_1)}{\mu_2-\xi}(\mathbf{q}-\mathbf{p}) \\
&\quad + \left(1 + \frac{\xi}{\mu_2-\xi}\right)\Big(\mathbf{p} + \mu_1(\mathbf{q}-\mathbf{p}) + \mu_2(\mathbf{b}-\mathbf{a}) - \mathbf{b}\Big) \\
&= \mathbf{p} + (1-\gamma)(\mathbf{q}-\mathbf{p}) + \frac{\xi(1-\gamma-\mu_1)}{\mu_2-\xi}(\mathbf{q}-\mathbf{p}) \\
&\quad + \left(1 + \frac{\xi}{\mu_2-\xi}\right)\Big(\xi(\mathbf{a}-\mathbf{b}) + (1-\gamma)(\mathbf{p}-\mathbf{q}) + \mu_1(\mathbf{q}-\mathbf{p}) + \mu_2(\mathbf{b}-\mathbf{a})\Big) \\
&= \mathbf{p} + (1-\gamma)(\mathbf{q}-\mathbf{p}) + \frac{\xi(1-\gamma-\mu_1)}{\mu_2-\xi}(\mathbf{q}-\mathbf{p}) + \left(1 + \frac{\xi}{\mu_2-\xi}\right)(1-\gamma-\mu_1)(\mathbf{p}-\mathbf{q}) + \\
&\qquad \left(\frac{\mu_2}{\mu_2-\xi}\right)(\mu_2-\xi)(\mathbf{b}-\mathbf{a}) \\
&= \mathbf{p} + \mu_1(\mathbf{q}-\mathbf{p}) + \mu_2(\mathbf{b}-\mathbf{a}) = \mathbf{z}.
\end{aligned}$$

To verify that the point

$$\begin{aligned}
&\left(\gamma - \frac{\xi(1-\gamma-\mu_1)}{\mu_2-\xi}\right)\mathbf{p} + \left(1 - \left(\gamma - \frac{\xi(1-\gamma-\mu_1)}{\mu_2-\xi}\right)\right)\mathbf{q} \\
&= \mathbf{p} + \left(1 - \left(\gamma - \frac{\xi(1-\gamma-\mu_1)}{\mu_2-\xi}\right)\right)(\mathbf{q}-\mathbf{p})
\end{aligned}$$

lies in the half-open interval $[\mathbf{y}, \mathbf{q})$, one must show $1 - \gamma \leqslant 1 - \left(\gamma - \frac{\xi(1-\gamma-\mu_1)}{\mu_2-\xi}\right) < 1$. In showing that the lower bound of the inequality holds, first notice that it immediately follows from (4.12) that

$$\mu_2 - \xi > \left(1 - \frac{\mu_1}{1-\gamma}\right)\left(\frac{(1-\gamma)(1-\partial(1-\xi))}{\partial\gamma}\right).$$

Recall that $0 < \gamma < 1$. It follows that $0 < 1-\gamma < 1$. This means $(1-\gamma)^2 > 0$. Therefore, $(1-\gamma)^2 + (1-\gamma) > 0$. Note that

$$\frac{1}{1-\gamma}\Big((1-\gamma)^2 + (1-\gamma)\Big) = (1-\gamma) + 1 > 0. \tag{4.13}$$

This implies that $\dfrac{1}{1-\gamma} > 0$. Combine inequalities (4.13) and (4.11) to get $0 \leqslant \dfrac{\mu_1}{1-\gamma} \leqslant 1$. It immediately follows that

$$0 \leqslant 1 - \frac{\mu_1}{1-\gamma} \leqslant 1. \tag{4.14}$$

Recall that $0 < \partial < 1$. Therefore, $\partial\gamma > 0$. Arguments similar to those above along with an inequality similar to (4.13) can be used to show that

$$\frac{1}{\partial\gamma} > 0. \tag{4.15}$$

Also, recall that $0 < \xi < 1$. It immediately follows that $0 < 1-\xi < 1$. This together with Corollary A.2 implies that $0 < \partial(1-\xi) < \partial < 1$. This means $0 < 1 - \partial(1-\xi) < 1$. Thus, $(1-\gamma)(1-\partial(1-\xi)) > 0$. This together with (4.15) implies that

$$\frac{(1-\gamma)(1-\partial(1-\xi))}{\partial\gamma} > 0. \tag{4.16}$$

The product of non-negative real numbers is itself a non-negative real number, so together (4.14) and (4.16) imply that

$$\left(1 - \frac{\mu_1}{1-\gamma}\right)\left(\frac{(1-\gamma)(1-\partial(1-\xi))}{\partial\gamma}\right) \geqslant 0.$$

This means $\mu_2 - \xi > 0$. Therefore,

$$\frac{1}{\mu_2 - \xi} > 0. \tag{4.17}$$

It follows from (4.11) that $1 - \gamma - \mu_1 \geqslant 0$. Therefore, $\xi(1-\gamma-\mu_1) \geqslant 0$. This together with (4.17) implies that

$$\frac{\xi(1-\gamma-\mu_1)}{\mu_2 - \xi} \geqslant 0.$$

Hence,

$$1-\left(\gamma-\frac{\xi\left(1-\gamma-\mu_1\right)}{\mu_2-\xi}\right)\geqslant 1-\gamma.$$

In verifying that 1 is a strict upper bound for $1-\left(\gamma-\dfrac{\xi\left(1-\gamma-\mu_1\right)}{\mu_2-\xi}\right)$, first notice that

$$\begin{aligned}\gamma\left(\mu_2-\xi\right)&>\gamma\left(\frac{1-\gamma-\mu_1}{1-\gamma}\right)\left(\frac{\left(1-\gamma\right)\left(1-\partial\left(1-\xi\right)\right)}{\partial\gamma}\right)\\&=\frac{\left(1-\gamma-\mu_1\right)\left(1-\partial\left(1-\xi\right)\right)}{\partial}.\end{aligned}$$

Again, recall that $0<\partial<1$. This means that $0<1-\partial<1$. Therefore,

$$\partial\left(\frac{1}{\partial}-1\right)=1-\partial>0.$$

This implies that $\dfrac{1}{\partial}-1>0$ and thus,

$$\frac{1}{\partial}>1. \tag{4.18}$$

It was shown above that $1-\gamma-\mu_1\geqslant 0$ and $0<1-\partial\left(1-\xi\right)<1$. This means that $\left(1-\gamma-\mu_1\right)\left(1-\partial\left(1-\xi\right)\right)\geqslant 0$. This together with (4.18) implies that

$$\frac{\left(1-\gamma-\mu_1\right)\left(1-\partial\left(1-\xi\right)\right)}{\partial}>\left(1-\gamma-\mu_1\right)\left(1-\partial\left(1-\xi\right)\right).$$

Therefore,

$$\gamma\left(\mu_2-\xi\right)>\left(1-\gamma-\mu_1\right)\left(1-\partial\left(1-\xi\right)\right).$$

It follows from Corollary A.2 that $\partial\left(1-\xi\right)<1-\xi$. This means $1-\partial\left(1-\xi\right)>1-\left(1-\xi\right)=\xi$. Therefore,

$$\gamma\left(\mu_2-\xi\right)>\xi\left(1-\gamma-\mu_1\right).$$

Recall from above that $\dfrac{1}{\mu_2-\xi}>0$. Hence,

$$\frac{\xi\left(1-\gamma-\mu_1\right)}{\mu_2-\xi}<\gamma.$$

It follows that

$$1-\left(\gamma-\frac{\xi\left(1-\gamma-\mu_1\right)}{\mu_2-\xi}\right)<1-\gamma+\gamma=1.$$

By Theorem 2.10.10, the half-open segment

$$\left[\left(\gamma-\frac{\xi\left(1-\gamma-\mu_1\right)}{\mu_2-\xi}\right)\mathbf{p}+\left(1-\left(\gamma-\frac{\xi\left(1-\gamma-\mu_1\right)}{\mu_2-\xi}\right)\right)\mathbf{q},\mathbf{b}\right),$$

on the ray emanating from the point

$$\left(\gamma-\frac{\xi\left(1-\gamma-\mu_1\right)}{\mu_2-\xi}\right)\mathbf{p}+\left(1-\left(\gamma-\frac{\xi\left(1-\gamma-\mu_1\right)}{\mu_2-\xi}\right)\right)\mathbf{q}$$

with direction $\mathbf{z}-\mathbf{b}$, is contained in relint(B). In other words,

$$\left(\gamma-\frac{\xi\left(1-\gamma-\mu_1\right)}{\mu_2-\xi}\right)\mathbf{p}+\left(1-\left(\gamma-\frac{\xi\left(1-\gamma-\mu_1\right)}{\mu_2-\xi}\right)\right)\mathbf{q}+\mu'\left(\mathbf{z}-\mathbf{b}\right)\in \operatorname{relint}(B),$$

for some $0\leqslant\mu'<\dfrac{\xi}{\mu_2-\xi}$. It follows that $\mathbf{z}\notin\operatorname{relint}(B)$.

By Corollary 2.10.12, the ray emanating from the point

$$\left(\gamma-\frac{\xi\left(1-\gamma-\mu_1\right)}{\mu_2-\xi}\right)\mathbf{p}+\left(1-\left(\gamma-\frac{\xi\left(1-\gamma-\mu_1\right)}{\mu_2-\xi}\right)\right)\mathbf{q}\in\operatorname{relint}(B)$$

with direction $\mathbf{z}-\mathbf{b}$ intersects the relbd(B) at exactly one point,

$$\mathbf{b}=\left(\gamma-\frac{\xi\left(1-\gamma-\mu_1\right)}{\mu_2-\xi}\right)\mathbf{p}+\left(1-\left(\gamma-\frac{\xi\left(1-\gamma-\mu_1\right)}{\mu_2-\xi}\right)\right)\mathbf{q}+\frac{\xi}{\mu_2-\xi}\left(\mathbf{z}-\mathbf{b}\right).$$

Clearly, $\mathbf{z}\neq\mathbf{b}$. Therefore, $\mathbf{z}\notin\operatorname{relbd}(B)$.

Thus, $\mathbf{z}\notin\operatorname{relint}(B)\cup\operatorname{relbd}(B)=B$, which is a contradiction.

Hence,

$$B_{\mathbf{b}}\cap B_{\mathbf{p}}\subseteq\operatorname{conv}\left\{\mathbf{p},\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q},\mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big(\mathbf{b}-\big(\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}\big)\Big)\right\}.$$

■

Consequently,

$$\begin{aligned}\mathbf{x}\in[\mathbf{b},\mathbf{p}]_B&=B_{\mathbf{b}}\cap B_{\mathbf{p}}\cap\operatorname{relbd}(B)\\&\subseteq\operatorname{conv}\left\{\mathbf{p},\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q},\mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big(\mathbf{b}-\big(\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}\big)\Big)\right\}.\end{aligned}$$

Notice that $\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}\notin B_{\mathbf{b}}\cap B_{\mathbf{p}}$. To see this, first observe that

$$\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}=\mathbf{a}+\partial\left(\mathbf{q}-\mathbf{a}\right)$$

$$= \mathbf{p} + (1-\gamma)(\mathbf{q}-\mathbf{p}) + (1-\xi)(\mathbf{a}-\mathbf{b}) + \partial\big((1-\xi)(\mathbf{b}-\mathbf{a}) + \gamma(\mathbf{q}-\mathbf{p})\big)$$
$$= \gamma(1-\partial)\mathbf{p} + (1-\gamma(1-\partial))\mathbf{q} + (1-\partial)(1-\xi)(\mathbf{a}-\mathbf{b}),$$

where $0 < (1-\partial)(1-\xi) < 1$ and $0 < \gamma(1-\partial) < 1$. This means that

$$(1-\partial)\mathbf{a} + \partial\mathbf{q} \notin \{(1-\Gamma)\mathbf{p} + \Gamma\mathbf{q} + \lambda'(\mathbf{b}-\mathbf{a}) \mid 0 \leqslant \Gamma \leqslant 1,\ \lambda' \geqslant 0\}.$$

This means

$$(1-\partial)\mathbf{a} + \partial\mathbf{q} \notin B \cap \{(1-\Gamma)\mathbf{p} + \Gamma\mathbf{q} + \lambda'(\mathbf{b}-\mathbf{a}) \mid 0 \leqslant \Gamma \leqslant 1,\ \lambda' \geqslant 0\} = B_{\mathbf{b}}.$$

Therefore, $(1-\partial)\mathbf{a} + \partial\mathbf{q} \notin B_{\mathbf{b}} \cap B_{\mathbf{p}}$.

Also, notice that

$$\mathbf{b} + \frac{1-\gamma}{\partial\gamma}\Big(\mathbf{b} - \big((1-\partial)\mathbf{a} + \partial\mathbf{q}\big)\Big) \notin B_{\mathbf{b}} \cap B_{\mathbf{p}}.$$

To see this, begin by noticing that the open line segment $(\mathbf{b}, (1-\partial)\mathbf{a} + \partial\mathbf{b}]$ belonging to the line passing through $\mathbf{b}$ and $(1-\partial)\mathbf{a} + \partial\mathbf{q}$ that intersects $(\mathbf{p}, \mathbf{q}) \subseteq \mathrm{relint}(B)$. Specifically,

$$\left(1 - \frac{\xi}{1-\partial(1-\xi)}\right)\mathbf{b} + \frac{\xi}{1-\partial(1-\xi)}\big((1-\partial)\mathbf{a} + \partial\mathbf{q}\big)$$
$$= \mathbf{b} + \frac{\xi}{1-\partial(1-\xi)}\big((1-\partial(1-\xi))(\mathbf{a}-\mathbf{b}) + \partial\gamma(\mathbf{q}-\mathbf{p})\big)$$
$$= \mathbf{p} + (1-\gamma)(\mathbf{q}-\mathbf{p}) + \xi(\mathbf{b}-\mathbf{a}) + \xi(\mathbf{a}-\mathbf{b}) + \frac{\xi\partial\gamma}{1-\partial(1-\xi)}(\mathbf{q}-\mathbf{p})$$
$$= \left(1 - \gamma\left(1 - \frac{\xi\partial}{1-\partial(1-\xi)}\right)\right)\mathbf{q} + \gamma\left(1 - \frac{\xi\partial}{1-\partial(1-\xi)}\right)\mathbf{p}$$

with

$$0 < \frac{\xi}{1-\partial(1-\xi)} < 1 \tag{4.19}$$

and

$$0 < \gamma\left(1 - \frac{\xi\partial}{1-\partial(1-\xi)}\right) < 1, \tag{4.20}$$

which means

$$\left(1 - \frac{\xi}{1-\partial(1-\xi)}\right)\mathbf{b} + \frac{\xi}{1-\partial(1-\xi)}\big((1-\partial)\mathbf{a} + \partial\mathbf{q}\big) \in \big(\mathbf{b}, (1-\partial)\mathbf{a} + \partial\mathbf{q}\big) \cap \big(\mathbf{p}, \mathbf{q}\big) \subseteq \mathrm{int}(K).$$

To see that (4.20) holds, first recall that $0 < \xi < 1$ and $0 < \partial < 1$. It immediately follows that

$$0 < \xi\partial < 1 - \partial + \xi\partial = 1 - \partial\left(1 - \xi\right). \tag{4.21}$$

Therefore, $\big(1 - \partial\left(1 - \xi\right)\big)^2 > 0$ and notice that

$$\frac{1}{1 - \partial\left(1 - \xi\right)}\big(1 - \partial\left(1 - \xi\right)\big)^2 = 1 - \partial\left(1 - \xi\right) > 0.$$

This together with (4.21) implies that

$$\frac{1}{1 - \partial\left(1 - \xi\right)} > 0. \tag{4.22}$$

Combine (4.21) with (4.22) to get $0 < \dfrac{\xi\partial}{1 - \partial\left(1 - \xi\right)} < 1$. This means that

$$0 < 1 - \frac{\xi\partial}{1 - \partial\left(1 - \xi\right)} < 1. \tag{4.23}$$

Recall that $0 < \gamma < 1$. By Corollary A.2 and (4.23),

$$0 < \gamma\left(1 - \frac{\xi\partial}{1 - \partial\left(1 - \xi\right)}\right) < \gamma < 1.$$

To see that (4.19) holds, first observe that $\partial\left(1 - \xi\right) < 1 - \xi$ by Corollary A.2 and therefore,

$$0 < \xi = 1 - \left(1 - \xi\right) < 1 - \partial\left(1 - \xi\right). \tag{4.24}$$

Combine (4.22) with (4.24) to get that

$$0 < \frac{\xi}{1 - \partial\left(1 - \xi\right)} < 1. \tag{4.25}$$

Now, recall that $\mathbf{b} \in \operatorname{relbd}(B)$. It follows from Corollary 2.10.12 that the ray emanating from the relative interior point

$$\left(1 - \frac{\xi}{1 - \partial\left(1 - \xi\right)}\right)\mathbf{b} + \frac{\xi}{1 - \partial\left(1 - \xi\right)}\Big(\left(1 - \partial\right)\mathbf{a} + \partial\mathbf{q}\Big) \in \Big(\mathbf{b}, \left(1 - \partial\right)\mathbf{a} + \partial\mathbf{q}\Big)$$

with direction $\mathbf{b} - \left(1 - \partial\right)\mathbf{a} + \partial\mathbf{q}$ intersects the relative boundary at exactly one point, $\mathbf{b}$. Note that

$$\mathbf{b} + \frac{1 - \gamma}{\partial\gamma}\Big(\mathbf{b} - \big(\left(1 - \partial\right)\mathbf{a} + \partial\mathbf{q}\big)\Big) \neq \mathbf{b},$$

since $\dfrac{1-\gamma}{\partial\gamma} > 0$. To see that this inequality holds, first deduce that $0 < \partial\gamma < 1$. This means that $0 < 1 - \partial\gamma < 1$. Notice that

$$\partial\gamma \left(\frac{1}{\partial\gamma} - 1 \right) = 1 - \partial\gamma > 0.$$

It follows that $\dfrac{1}{\partial\gamma} > 1$ and therefore,

$$\frac{1-\gamma}{\partial\gamma} > 1 - \gamma > 0. \tag{4.26}$$

Hence, the point $\mathbf{b} + \dfrac{1-\gamma}{\partial\gamma}\Big(\mathbf{b} - \big((1-\partial)\,\mathbf{a} + \partial\mathbf{q}\big)\Big)$, on the ray emanating from

$$\left(1 - \frac{\xi}{1-\partial\,(1-\xi)}\right)\mathbf{b} + \frac{\xi}{1-\partial\,(1-\xi)}\Big(\,(1-\partial)\,\mathbf{a} + \partial\mathbf{q}\Big) \in \Big(\mathbf{b}, (1-\partial)\,\mathbf{a} + \partial\mathbf{q}\Big)$$

with direction $\mathbf{b} - (1-\partial)\,\mathbf{a} + \partial\mathbf{q}$, does not belong to $\mathrm{relbd}(B)$.

Moreover, it follows from Theorem 2.10.10 that

$$\left(\mathbf{b}, \left(1 - \frac{\xi}{1-\partial\,(1-\xi)}\right)\mathbf{b} + \frac{\xi}{1-\partial\,(1-\xi)}\Big(\,(1-\partial)\,\mathbf{a} + \partial\mathbf{q}\Big)\right] \subseteq \mathrm{relint}(B).$$

In other words, the intersection between the ray emanating from

$$\left(1 - \frac{\xi}{1-\partial\,(1-\xi)}\right)\mathbf{b} + \frac{\xi}{1-\partial\,(1-\xi)}\Big(\,(1-\partial)\,\mathbf{a} + \partial\mathbf{q}\Big)$$

with direction $\mathbf{b} - (1-\partial)\,\mathbf{a} + \partial\mathbf{q}$ and $\mathrm{relint}(B)$ is the half-open interval

$$\left(\mathbf{b}, \left(1 - \frac{\xi}{1-\partial\,(1-\xi)}\right)\mathbf{b} + \frac{\xi}{1-\partial\,(1-\xi)}\Big(\,(1-\partial)\,\mathbf{a} + \partial\mathbf{q}\Big)\right].$$

Every element in this half-open line segment has the form $\mathbf{b} + \lambda\big(\,(1-\partial)\,\mathbf{a} + \partial\mathbf{q} - \mathbf{b}\big)$, for some $0 < \lambda \leqslant \dfrac{\xi}{1-\partial\,(1-\xi)}$. It follows from (4.26) that $-\dfrac{1-\gamma}{\partial\gamma} < 0$. Therefore,

$$\mathbf{b} - \frac{1-\gamma}{\partial\gamma}\Big(\,(1-\partial)\,\mathbf{a} + \partial\mathbf{q} - \mathbf{b}\Big) \notin \left(\mathbf{b}, \left(1 - \frac{\xi}{1-\partial\,(1-\xi)}\right)\mathbf{b} + \frac{\xi}{1-\partial\,(1-\xi)}\Big(\,(1-\partial)\,\mathbf{a} + \partial\mathbf{q}\Big)\right].$$

Hence,

$$\mathbf{b} + \frac{1-\gamma}{\partial\gamma}\Big(\mathbf{b} - \big(\,(1-\partial)\,\mathbf{a} + \partial\mathbf{q}\big)\Big) \notin \mathrm{relint}(B).$$

It follows that

$$\mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big(\mathbf{b}-\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}\big)\Big)\notin \operatorname{relbd}(B)\cup\operatorname{relint}(B)=B.$$

Therefore,

$$\mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big(\mathbf{b}-\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}\big)\Big)\notin B_{\mathbf{b}}\cap B_{\mathbf{p}}.$$

Accordingly, there exists real numbers $0<\sigma_1\leqslant 1$ and $0\leqslant\sigma_2\leqslant 1-\sigma_1<1$ such that

$$\mathbf{x}=\sigma_1\mathbf{p}+\sigma_2\left((1-\partial)\,\mathbf{a}+\partial\mathbf{q}\right)+(1-\sigma_1-\sigma_2)\left(\mathbf{b}-\frac{1-\gamma}{\partial\gamma}\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\big)\right),$$

for any $\mathbf{x}\in B_{\mathbf{b}}\cap B_{\mathbf{p}}\cap\operatorname{relbd}(B)=[\mathbf{b},\mathbf{p}]_B$.

To show that $\mathbf{x}\in B_{\mathbf{p}}\cap B_{\mathbf{b}}\cap\operatorname{relbd}(B)=[\mathbf{b},\mathbf{p}]_B$ is illuminated by any direction $(1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}$, for some $0<\partial<1$, begin by considering the case where $\sigma_1=1$ and $\sigma_2=0$. In this case, $\mathbf{x}=\mathbf{p}$. Recall that $\mathbf{p}$ is a smooth point on the $\operatorname{relbd}(B)$. This means the supporting line of B at $\mathbf{p}$, ℓ, is unique. Notice that the line $\{\mathbf{p}+\lambda\left((1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\right)\mid\lambda\in\mathbb{R}\}$ is parallel to the line passing through the points $\mathbf{b}$ and $(1-\partial)\,\mathbf{a}+\partial\mathbf{q}$. Recall that the line passing through the points $\mathbf{b}$ and $(1-\partial)\,\mathbf{a}+\partial\mathbf{q}$ intersects the line ℓ at the point $\mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big(\mathbf{b}-\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}\big)\Big)$ and therefore, ℓ is not parallel to the line passing through the points $\mathbf{b}$ and $(1-\partial)\,\mathbf{a}+\partial\mathbf{q}$. Hence, ℓ is not parallel to the line $\{\mathbf{p}+\lambda\left((1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\right)\mid\lambda\in\mathbb{R}\}$. It follows from Proposition 2.10.1.4 that

$$\{\mathbf{p}+\lambda\left((1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\right)\mid\lambda\in\mathbb{R}\}\cap\operatorname{relint}(B)\neq\varnothing.$$

This means that either the ray emanating from $\mathbf{p}$ with direction $(1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}$ intersects $\operatorname{relint}(B)$ or the ray emanating from $\mathbf{p}$ with direction $\mathbf{b}-\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}\big)$ intersects $\operatorname{relint}(B)$. Recall that B lies between the lines ℓ and ℓ'. This implies that the ray which passes through the region between the lines ℓ and ℓ' will intersect $\operatorname{relint}(B)$. Observe that the point $\mathbf{p}+\frac{1-\gamma}{\partial\gamma}\Big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\Big)$ on the ray emanating from $\mathbf{p}$ with direction $(1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}$ intersects the line ℓ_*, which lies in the region between ℓ and ℓ':

$$\mathbf{p}+\frac{1-\gamma}{\partial\gamma}\Big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\Big)$$

$$= \mathbf{p} + (1-\gamma)(\mathbf{q}-\mathbf{p}) + \frac{1-\partial(1-\xi)(1-\gamma)}{\partial\gamma}(\mathbf{a}-\mathbf{b})$$
$$= \mathbf{y} + \frac{1-\partial(1-\xi)(1-\gamma)}{\partial\gamma}(\mathbf{a}-\mathbf{b}) \in \ell_*.$$

Therefore, the direction $(1-\partial)\mathbf{a} + \partial\mathbf{q} - \mathbf{b}$ illuminates $\mathbf{p}$. This means that there exists an element $\mathbf{d}$, which belongs to relint(B) and the ray emanating from $\mathbf{p}$ with direction $(1-\partial)\mathbf{a} + \partial\mathbf{q} - \mathbf{b}$.

Note that by Theorem 2.10.10, $(\mathbf{p}, \mathbf{d}] \subseteq \text{relint}(B)$. Let $\mathbf{d}' \in (\mathbf{p}, \mathbf{d}]$ be chosen so that

$$\mathbf{d}' - \mathbf{p} = \eta\big((1-\partial)\mathbf{a} + \partial\mathbf{q} - \mathbf{b}\big),$$

for some $0 < \eta < 1$. It follows from Theorem 2.10.10 that

$$\big((1-\partial)\mathbf{a} + \partial\mathbf{q}, \mathbf{d}'\big] \subseteq \text{relint}(B).$$

Now, consider the case where $0 < \sigma_1 < 1$ and $0 < \sigma_2 \leqslant 1 - \sigma_1$. It follows that

$$0 \leqslant 1 - \sigma_1 - \sigma_2 \leqslant 1 - \sigma_1 < 1. \tag{4.27}$$

The point

$$\mathbf{x} + \left((1-\sigma_1-\sigma_2) + \frac{(1-\sigma_1-\sigma_2)(1-\gamma)}{\partial\gamma} + \eta\sigma_1\right)\big((1-\partial)\mathbf{a} + \partial\mathbf{q} - \mathbf{b}\big)$$

belongs to both the ray emanating from $\mathbf{x}$ with direction $(1-\partial)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}$ and the half-open line segment $\big((1-\partial)\mathbf{a} + \partial\mathbf{q}, \mathbf{d}'\big] \subseteq \text{relint}(B)$.

To see this, first observe that

$$(1-\sigma_1-\sigma_2) + \frac{(1-\sigma_1-\sigma_2)(1-\gamma)}{\partial\gamma} + \eta\sigma_1 > 0.$$

This follows from the inequalities (4.27), (4.26), $1 < \sigma_1 \leqslant 1$ and $0 < \eta < 1$.

Then, notice that

$$\mathbf{x} + \left((1-\sigma_1-\sigma_2) + \frac{(1-\sigma_1-\sigma_2)(1-\gamma)}{\partial\gamma} + \eta\sigma_1\right)\big((1-\partial)\mathbf{a} + \partial\mathbf{q} - \mathbf{b}\big)$$
$$= \sigma_1\mathbf{p} + \sigma_2\big((1-\partial)\mathbf{a} + \partial\mathbf{q}\big) + (1-\sigma_1+\sigma_2)\left(\mathbf{b} - \frac{1-\gamma}{\partial\gamma}\big((1-\partial)\mathbf{a} + \partial\mathbf{q} - \mathbf{b}\big)\right) +$$

$$\left((1-\sigma_1-\sigma_2)+\frac{(1-\sigma_1-\sigma_2)(1-\gamma)}{\partial\gamma}+\eta\sigma_1\right)\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\big)$$
$$=\sigma_1\mathbf{p}+(1-\sigma_1)\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}\big)++\eta\sigma_1\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\big)$$
$$=\sigma_1\mathbf{d}'+(1-\sigma_1)\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}\big)\in\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q},\mathbf{d}'\big],$$

since $0<\sigma_1<1$.

Hence, the directions $(1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}$ illuminate the closed curve $B_{\mathbf{b}}\cap B_{\mathbf{p}}\cap\text{relbd}(B)=[\mathbf{b},\mathbf{p}]_B$, for any $0<\partial<1$. Recall that the directions $(1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}$ and $R_{x_3}(\theta)(\mathbf{a}-\mathbf{b})$ are parallel. This means $R_{x_3}(\theta)(\mathbf{a}-\mathbf{b})$ illuminates the closed curve $B_{\mathbf{b}}\cap B_{\mathbf{p}}\cap\text{relbd}(B)=[\mathbf{b},\mathbf{p}]_B$, for any $0<\theta<\alpha$.

Similar arguments can be used to show that for any angle $0<\Upsilon<\beta$ there exists some scalar $0<\partial'<1$ such that the vector $R_{x_3}(-\Upsilon)(\mathbf{b}-\mathbf{a})$ is parallel to $(1-\partial')(\mathbf{b}-\mathbf{a})+\partial'(\mathbf{q}-\mathbf{a})$ and ultimately, that the directions $R_{x_3}(-\Upsilon)(\mathbf{b}-\mathbf{a})$ illuminate the closed arc $B_{\mathbf{a}}\cap B_{\mathbf{p}}\cap\text{relbd}(B)=[\mathbf{p},\mathbf{a}]_B$ for any $0<\Upsilon<\beta$.

It follows that that any of the directions $R_{x_3}(\theta)(\mathbf{a}-\mathbf{b})$ together with any of the directions $R_{x_3}(\Upsilon)(\mathbf{b}-\mathbf{a})$, for any $0<\theta<\alpha$ and $0<\Upsilon<\beta$, will illuminate the closed arc $B_{\mathbf{p}}\cap\text{relbd}(B)=[\mathbf{b},\mathbf{a}]_B$ of the closed curve relbd(B) in the x_1x_2-plane.

For the sake of simplicity, choose the specific directions $\frac{1}{2}(\mathbf{q}+\mathbf{a})-\mathbf{b}$ and $\frac{1}{2}(\mathbf{q}+\mathbf{b})-\mathbf{a}$.

Case 2: Suppose that either $\mathbf{x}\in\Big(W\backslash\text{relbd}(B)\Big)\cap H_+$ or $\mathbf{x}\in\text{relint}(B)_+=\text{Pr}^{-1}(\text{relint}(B))\cap\text{bd}(K)\cap H_+$.

Recall from above that $\mathbf{x}\notin\text{slab}\left[\text{Pr}^{-1}(\ell_*),\text{Pr}^{-1}(\ell')\right]$. Note that slab$\left[\text{Pr}^{-1}(\ell_*),\text{Pr}^{-1}(\ell')\right]$ can be expressed as $\left\{\mathbf{z}\in\mathbb{E}^3\mid\text{Pr}(\mathbf{z})\in\left(1-\widetilde{\Lambda}\right)\ell+\widetilde{\Lambda}\ell',\text{ for }\Lambda\leqslant\widetilde{\Lambda}\leqslant 1\right\}$. This means $\text{Pr}(\mathbf{x})\notin\left(1-\widetilde{\Lambda}\right)\ell+\widetilde{\Lambda}\ell'$, for any $\Lambda\leqslant\widetilde{\Lambda}\leqslant 1$. Therefore, $\text{Pr}(\mathbf{x})\notin\text{slab}\left[\text{Pr}^{-1}(\ell_*),\text{Pr}^{-1}(\ell')\right]$. Again, recall that $B_{\mathbf{q}}\subseteq\text{slab}\left[\text{Pr}^{-1}(\ell_*),\text{Pr}^{-1}(\ell')\right]$. Hence, $\text{Pr}(\mathbf{x})\notin B_{\mathbf{q}}$. It follows that $\text{Pr}(\mathbf{x})\in B\backslash B_{\mathbf{q}}\subseteq B_{\mathbf{p}}$.

Suppose, furthermore, that $\text{Pr}(\mathbf{x})\in B_{\mathbf{b}}$.

It was shown in Case 1 that

$$B_{\mathbf{b}}\cap B_{\mathbf{p}}\subseteq\text{conv}\left\{\mathbf{p},(1-\partial)\,\mathbf{a}+\partial\mathbf{q},\mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big(\mathbf{b}-\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}\big)\Big)\right\}$$

and that

$$(1-\partial)\,\mathbf{a}+\partial\mathbf{q}, \mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big(\mathbf{b}-\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}\big)\Big)\notin B_{\mathbf{b}}\cap B_{\mathbf{p}}.$$

This means $\Pr(x)$ can be written as

$$\sigma_1'\mathbf{p}+\sigma_2'\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}\big)+(1-\sigma_1'-\sigma_2')\left(\mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\Big)\right),$$

for some $0<\sigma_1'\leqslant 1$ and $0\leqslant\sigma_2'<1-\sigma_1'$.

However, it is important in this case to notice that

$$\sigma_1'\mathbf{p}+\sigma_2'\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}\big)+(1-\sigma_1'-\sigma_2')\left(\mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\Big)\right)\notin B_{\mathbf{b}}\cap B_{\mathbf{p}},$$

for

$$1\geqslant\sigma_1'\geqslant\frac{\partial\gamma\xi}{(1-\gamma)\,(1-\partial\,(1-\xi))+\partial\gamma\xi}$$

and

$$\frac{1-\gamma}{\partial\gamma+1-\gamma}<\sigma_2'<1-\sigma_1'.$$

To see this, observe that

$$\begin{aligned}
&\sigma_1'\mathbf{p}+\sigma_2'\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}\big)+(1-\sigma_1'-\sigma_2')\left(\mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\Big)\right)\\
&=\mathbf{p}+\left[(1-\sigma_1')\,(1-\gamma)+\left(\sigma_2'-(1-\sigma_1'-\sigma_2')\,\frac{1-\gamma}{\partial\gamma}\right)\partial\gamma\right](\mathbf{q}-\mathbf{p})+\\
&\quad\left[\left(\sigma_2'-(1-\sigma_1'-\sigma_2')\,\frac{1-\gamma}{\partial\gamma}\right)(1-\partial\,(1-\xi))-(1-\sigma_1')\,\xi\right](\mathbf{a}-\mathbf{b})
\end{aligned}$$

and observe that

$$\begin{aligned}
&\left(\sigma_2'-(1-\sigma_1'-\sigma_2')\,\frac{1-\gamma}{\partial\gamma}\right)(1-\partial\,(1-\xi))-(1-\sigma_1')\,\xi\\
&=\sigma_2'\,(1-\partial\,(1-\xi))\left(1+\frac{1-\gamma}{\partial\gamma}\right)-(1-\sigma_1')\left(\xi+(1-\partial\,(1-\xi))\,\frac{1-\gamma}{\partial\gamma}\right)\\
&>\left(\frac{1-\gamma}{\partial\gamma+1-\gamma}\right)\left(\frac{\partial\gamma+1-\gamma}{\partial\gamma}\right)(1-\partial\,(1-\xi))-(1-\sigma_1')\left(\xi+(1-\partial\,(1-\xi))\,\frac{1-\gamma}{\partial\gamma}\right)\\
&=\sigma_1'\left(\frac{(1-\partial\,(1-\xi))\,(1-\gamma)+\partial\gamma\xi}{\partial\gamma}\right)-\xi\\
&\geqslant\left(\frac{\partial\gamma\xi}{(1-\gamma)\,(1-\partial\,(1-\xi))+\partial\gamma\xi}\right)\left(\frac{(1-\partial\,(1-\xi))\,(1-\gamma)+\partial\gamma\xi}{\partial\gamma}\right)-\xi=\xi-\xi=0.
\end{aligned}$$

Moreover,

$$(1-\sigma_1')(1-\gamma)+\left(\sigma_2'-(1-\sigma_1'-\sigma_2')\frac{1-\gamma}{\partial\gamma}\right)\partial\gamma=\sigma_2'(\partial\gamma+1-\gamma)$$
$$>\frac{1-\gamma}{\partial\gamma+1-\gamma}(\partial\gamma+1-\gamma)=1-\gamma>0$$

and it follows from Corollary A.2 that

$$(1-\sigma_1')(1-\gamma)+\left(\sigma_2'-(1-\sigma_1'-\sigma_2')\frac{1-\gamma}{\partial\gamma}\right)\partial\gamma=\sigma_2'(\partial\gamma+1-\gamma)$$
$$\leqslant\sigma_2'\leqslant 1-\sigma_1'<1.$$

In particular, this implies that

$$\sigma_1'\mathbf{p}+\sigma_2'\big((1-\partial)\mathbf{a}+\partial\mathbf{q}\big)+(1-\sigma_1'-\sigma_2')\left(\mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big((1-\partial)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\Big)\right)\in B_{\mathbf{a}}.$$

Recall from above that $B_{\mathbf{a}}=(B\backslash B_{\mathbf{b}})\cup[\mathbf{p},\mathbf{q}]$ Therefore,

$$\sigma_1'\mathbf{p}+\sigma_2'\big((1-\partial)\mathbf{a}+\partial\mathbf{q}\big)+(1-\sigma_1'-\sigma_2')\left(\mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big((1-\partial)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\Big)\right)\notin B_{\mathbf{b}}.$$

And thus,

$$\sigma_1'\mathbf{p}+\sigma_2'\big((1-\partial)\mathbf{a}+\partial\mathbf{q}\big)+(1-\sigma_1'-\sigma_2')\left(\mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big((1-\partial)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\Big)\right)\notin B_{\mathbf{b}}\cap B_{\mathbf{p}}.$$

This means

$$\Pr(\mathbf{x})=\sigma_1'\mathbf{p}+\sigma_2'\big((1-\partial)\mathbf{a}+\partial\mathbf{q}\big)+(1-\sigma_1'-\sigma_2')\left(\mathbf{b}+\frac{1-\gamma}{\partial\gamma}\Big((1-\partial)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\Big)\right),$$

for some

$$0<\sigma_1'\leqslant\frac{\partial\gamma\xi}{(1-\gamma)(1-\partial(1-\xi))+\partial\gamma\xi}$$

and

$$0<\sigma_2'\leqslant\frac{1-\gamma}{\partial\gamma+1-\gamma}.$$

In verifying the upper bound of σ_1' is well defined, first notice that

$$\big((1-\gamma)(1-\partial(1-\xi))+\partial\gamma\xi\big)\left(\frac{\partial\gamma\xi}{(1-\gamma)(1-\partial(1-\xi))+\partial\gamma\xi}\right)=\partial\gamma\xi$$

and

$$\big((1-\gamma)(1-\partial(1-\xi))+\partial\gamma\xi\big)\left(1-\frac{\partial\gamma\xi}{(1-\gamma)(1-\partial(1-\xi))+\partial\gamma\xi}\right)=(1-\gamma)(1-\partial(1-\xi)).$$

Recall from (4.21) that $1-\partial(1-\xi)>0$. It follows immediately from $0<\gamma<1$ that $1-\gamma>0$. Therefore,

$$(1-\partial(1-\xi))(1-\gamma)>0. \tag{4.28}$$

Also, recall that $0<\partial,\xi<1$. This means that $\partial\gamma\xi>0$. Thus,

$$(1-\partial(1-\xi))(1-\gamma)+\partial\gamma\xi>0.$$

Hence,

$$0<\frac{\partial\gamma\xi}{(1-\gamma)(1-\partial(1-\xi))+\partial\gamma\xi}<1.$$

To verify the upper bound of σ_2' is well defined, observe that

$$\big(\partial\gamma+1-\gamma\big)\left(\frac{1-\gamma}{\partial\gamma+1-\gamma}\right)=1-\gamma>0$$

and

$$\big(\partial\gamma+1-\gamma\big)\left(1-\sigma_1'-\frac{1-\gamma}{\partial\gamma+1-\gamma}\right)=(\partial\gamma+1-\gamma)(1-\sigma_1')-(1-\gamma).$$

Again, recall that $0<\gamma,\partial<1$. Consequently, $\partial\gamma>0$ and $1-\gamma>0$. This means $\partial\gamma+1-\gamma>0$ and thus,

$$\frac{1-\gamma}{\partial\gamma+1-\gamma}>0.$$

Use the improved upper bound on σ_1' to get

$$\begin{aligned}
&(\partial\gamma+1-\gamma)(1-\sigma_1')-(1-\gamma)\\
&\geqslant(\partial\gamma+1-\gamma)\left(\frac{(1-\gamma)(1-\partial(1-\xi))}{(1-\gamma)(1-\partial(1-\xi))+\partial\gamma\xi}\right)-(1-\gamma)\\
&=(1-\gamma)\left(\frac{(\partial\gamma+1-\gamma)(1-\partial(1-\xi))-(1-\gamma)(1-\partial(1-\xi))-\partial\gamma\xi}{(1-\gamma)(1-\partial(1-\xi))+\partial\gamma\xi}\right)\\
&=\frac{(1-\gamma)\partial\gamma(1-\xi)(1-\partial)}{(1-\gamma)(1-\partial(1-\xi))+\partial\gamma\xi}
\end{aligned}$$

Notice that

$$((1-\gamma)(1-\partial(1-\xi))+\partial\gamma\xi)\left(\frac{(1-\gamma)\partial\gamma(1-\xi)(1-\partial)}{(1-\gamma)(1-\partial(1-\xi))+\partial\gamma\xi}\right)=(1-\gamma)\partial\gamma(1-\xi)(1-\partial),$$

where $(1-\gamma)(1-\partial(1-\xi))+\partial\gamma\xi>0$ from (4.28) and $(1-\gamma)\partial\gamma(1-\xi)(1-\partial)>0$ can be similarly shown. Thus,

$$\frac{(1-\gamma)\partial\gamma(1-\xi)(1-\partial)}{(1-\gamma)(1-\partial(1-\xi))+\partial\gamma\xi}>0,$$

which implies that $(\partial\gamma+1-\gamma)(1-\sigma_1')-(1-\gamma)>0$. Hence,

$$\frac{1-\gamma}{\partial\gamma+1-\gamma}<1-\sigma_1'.$$

It follows from Case 1 that

$$\begin{aligned}&\Pr(\mathbf{x})+\left((1-\sigma_1'-\sigma_2')\left(1+\frac{1-\gamma}{\partial\gamma}\right)+\eta\sigma_1'\right)\big((1-\partial)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\big)\\&=\sigma_1'\mathbf{d}'+(1-\sigma_1')((1-\partial)\mathbf{a}+\partial\mathbf{q})\in\big((1-\partial)\mathbf{a}+\partial\mathbf{q},\mathbf{d}'\big]\end{aligned}$$

Recall that $\mathcal{T}=\max\{\|\mathbf{k}^+-\mathbf{k}^-\|\mid\mathbf{k}\in K\}$. It follows that

$$\mathbf{x}=\Pr(\mathbf{x})+\rho\cdot\frac{\mathcal{T}}{2}\mathbf{e}_3,$$

for some $0<\rho\leqslant 1$. Let $\zeta=(1-\sigma_1'-\sigma_2')\left(1+\dfrac{1-\gamma}{\partial\gamma}\right)+\eta\sigma_1'$.

Now, observe that

$$\begin{aligned}&\mathbf{x}+\left(\frac{\rho}{2}\right)\left[\left(1-\frac{\xi}{1-\partial(1-\xi)}\right)\big((1-\partial)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\big)-\mathcal{T}\mathbf{e}_3\right]\\&=\Pr(\mathbf{x})+\rho\cdot\frac{\mathcal{T}}{2}\mathbf{e}_3-\rho\cdot\frac{\mathcal{T}}{2}\mathbf{e}_3+\left(\frac{\rho}{2}\right)\left(1-\frac{\xi}{1-\partial(1-\xi)}\right)\big((1-\partial)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\big)\\&=\Pr(\mathbf{x})+\left(\frac{\rho}{2}\right)\left(1-\frac{\xi}{1-\partial(1-\xi)}\right)\left(\frac{1}{\zeta}\right)\Big(\sigma_1'\mathbf{d}'+(1-\sigma_1')\big((1-\partial)\mathbf{a}+\partial\mathbf{q}\big)-\Pr(\mathbf{x})\Big)\\&=\left(1-\left(\frac{\rho}{2}\right)\left(1-\frac{\xi}{1-\partial(1-\xi)}\right)\left(\frac{1}{\zeta}\right)\right)\Pr(\mathbf{x})+\\&\quad\left(\frac{\rho}{2}\right)\left(1-\frac{\xi}{1-\partial(1-\xi)}\right)\left(\frac{1}{\zeta}\right)\Big(\sigma_1'\mathbf{d}'+(1-\sigma_1')\big((1-\partial)\mathbf{a}+\partial\mathbf{q}\big)\Big).\end{aligned}$$

It will be shown that

$$\mathbf{x}+\left(\frac{\rho}{2}\right)\left[\left(1-\frac{\xi}{1-\partial(1-\xi)}\right)\big((1-\partial)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\big)-\mathcal{T}\mathbf{e}_3\right]$$

$$\in \left(\Pr(\mathbf{x}), \sigma_1' \mathbf{d}' + (1-\sigma_1')((1-\partial)\mathbf{a} + \partial \mathbf{q}) \right].$$

First, notice that

$$\left((1-\gamma)(1-\partial(1-\xi)) + \partial\gamma\xi \right) \left(\frac{\partial\gamma + 1 - \gamma}{(1-\gamma)(1-\partial(1-\xi)) + \partial\gamma\xi} \right) = \partial\gamma + 1 - \gamma.$$

Recall from (4.28) that $(1-\gamma)(1-\partial(1-\xi)) + \partial\gamma\xi > 0$. Also, recall from above that $\partial\gamma + 1 - \gamma > 0$. Hence,

$$\frac{\partial\gamma + 1 - \gamma}{(1-\partial(1-\xi))(1-\gamma) + \partial\gamma\xi} > 0. \tag{4.29}$$

Then, observe that

$$\zeta = (1-\sigma_1'-\sigma_2')\left(1 + \frac{1-\gamma}{\partial\gamma}\right) + \eta\sigma_1' > -\sigma_2'\left(1 + \frac{1-\gamma}{\partial\gamma}\right) + (1-\sigma_1')\left(1 + \frac{1-\gamma}{\partial\gamma}\right).$$

Use the improved upper bound on σ_1' to get

$$\begin{aligned} &\geqslant -\left(\frac{1-\gamma}{\partial\gamma + 1 - \gamma}\right)\left(\frac{\partial\gamma + 1 - \gamma}{\partial\gamma}\right) + (1-\sigma_1')\left(1 + \frac{1-\gamma}{\partial\gamma}\right) \\ &= 1 - \sigma_1'\left(\frac{\partial\gamma + 1 - \gamma}{\partial\gamma}\right) \end{aligned}$$

Now, use the improved upper bound on σ_2' to get

$$\begin{aligned} &\geqslant 1 - \left(\frac{\partial\gamma\xi}{(1-\gamma)(1-\partial(1-\xi)) + \partial\gamma\xi}\right)\left(\frac{\partial\gamma + 1 - \gamma}{\partial\gamma}\right) \\ &= 1 - \left(\frac{\xi}{1-\partial(1-\xi)}\right)\left(\frac{\partial\gamma + 1 - \gamma}{(1-\gamma)(1-\partial(1-\xi)) + \partial\gamma\xi}\right) \end{aligned}$$

Then, by (4.25), (4.29) and Proposition A.1

$$\geqslant 1 - \frac{\xi}{1-\partial(1-\xi)}.$$

Recall from (4.25) that

$$0 < \frac{\xi}{1-\partial(1-\xi)} < 1.$$

It immediately follows that

$$\zeta > 1 - \frac{\xi}{1-\partial(1-\xi)} > 0. \tag{4.30}$$

Also, notice that

$$\zeta\left(\frac{1}{\zeta}\right)=1>0.$$

This together with (4.30) implies that $\frac{1}{\zeta}>0$. Multiply (4.30) by $\frac{1}{\zeta}$ to get

$$0<\left(1-\frac{\xi}{1-\partial\left(1-\xi\right)}\right)\left(\frac{1}{\zeta}\right)<1. \tag{4.31}$$

Recall that $0<\rho\leqslant 1$. It immediately follows that $0<\frac{\rho}{2}\leqslant\frac{1}{2}<1$. Combine this with (4.31) to get

$$0<\left(\frac{\rho}{2}\right)\left(1-\frac{\xi}{1-\partial\left(1-\xi\right)}\right)\left(\frac{1}{\zeta}\right)<1.$$

It follows that

$$\begin{aligned}
&\mathbf{x}+\left(\frac{\rho}{2}\right)\left[\left(1-\frac{\xi}{1-\partial\left(1-\xi\right)}\right)\big(\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\big)-\mathcal{T}\mathbf{e}_3\right]\\
&=\left(1-\left(\frac{\rho}{2}\right)\left(1-\frac{\xi}{1-\partial\left(1-\xi\right)}\right)\left(\frac{1}{\zeta}\right)\right)\Pr\left(\mathbf{x}\right)\\
&\quad+\left(\frac{\rho}{2}\right)\left(1-\frac{\xi}{1-\partial\left(1-\xi\right)}\right)\left(\frac{1}{\zeta}\right)\Big(\sigma_1'\mathbf{d}'+\left(1-\sigma_1'\right)\big(\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}\big)\Big)\\
&\in\big(\Pr\left(\mathbf{x}\right),\sigma_1'\mathbf{d}'+\left(1-\sigma_1'\right)\left(\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}\right)\big]
\end{aligned}$$

Recall from Case 1 that any element in $\big(\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q},\mathbf{d}'\big]$ belongs to relint(B). It follows from this and Theorem 2.10.10 that

$$\big(\Pr\left(\mathbf{x}\right),\sigma_1'\mathbf{d}'+\left(1-\sigma_1'\right)\left(\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}\right)\big]\subseteq\operatorname{relint}(B).$$

Thus,

$$\mathbf{x}+\left(\frac{\rho}{2}\right)\left[\left(1-\frac{\xi}{1-\partial\left(1-\xi\right)}\right)\big(\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\big)-\mathcal{T}\mathbf{e}_3\right]\in\operatorname{relint}(B)\subseteq\operatorname{int}(K).$$

Hence, the direction

$$\left(1-\frac{\xi}{1-\partial\left(1-\xi\right)}\right)\big(\left(1-\partial\right)\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\big)-\mathcal{T}\mathbf{e}_3$$

illuminates any $\mathbf{x}\in\Big(W\backslash\operatorname{relbd}(B)\Big)\cap H_{+}$ or $\mathbf{x}\in\operatorname{relint}(B)_{+}$ as long as $\Pr\left(\mathbf{x}\right)\in B_{\mathbf{b}}$.

A very similar method can be used to show that the direction

$$\left(1-\frac{1-\xi}{1-\partial'\xi}\right)\big((1-\partial')\,\mathbf{b}+\partial'\mathbf{q}-\mathbf{a}\big)-\mathcal{T}\mathbf{e}_3$$

illuminates any $\mathbf{x}\in\Big(W\backslash\mathrm{relbd}(B)\Big)\cap H_{+}$ or $\mathbf{x}\in\mathrm{relint}(B)_{+}$ where $\mathrm{Pr}(\mathbf{x})\in B_{\mathbf{a}}$.

The case where $\mathbf{x}\in\mathrm{relint}(B)_{-}$ or $\mathbf{x}\in\Big(W\backslash\mathrm{relbd}(B)\Big)\cap H_{-}$ will follow similarly with the directions

$$\left(1-\frac{\xi}{1-\partial\,(1-\xi)}\right)\big((1-\partial)\,\mathbf{a}+\partial\mathbf{q}-\mathbf{b}\big)+\mathcal{T}\mathbf{e}_3$$

and

$$\left(1-\frac{1-\xi}{1-\partial'\xi}\right)\big((1-\partial')\,\mathbf{b}+\partial'\mathbf{q}-\mathbf{a}\big)+\mathcal{T}\mathbf{e}_3.$$

For the sake of simplicity, choose the specific directions

$$\left(1-\frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}\,(\mathbf{q}+\mathbf{a})-\mathbf{b}\right)\pm\mathcal{T}\mathbf{e}_3$$

and

$$\left(1-\frac{2\,(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}\,(\mathbf{q}+\mathbf{b})-\mathbf{a}\right)\pm\mathcal{T}\mathbf{e}_3.$$

Thus, the seven directions $\mathbf{p}-\mathbf{q}$, $\frac{1}{2}\,(\mathbf{q}+\mathbf{a})-\mathbf{b}$, $\frac{1}{2}\,(\mathbf{q}+\mathbf{b})-\mathbf{a}$, $\left(1-\frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}\,(\mathbf{q}+\mathbf{a})-\mathbf{b}\right)\pm\mathcal{T}\mathbf{e}_3$ and $\left(1-\frac{2(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}\,(\mathbf{q}+\mathbf{b})-\mathbf{a}\right)\pm\mathcal{T}\mathbf{e}_3$ will illuminate K. ■

Proposition 4.2.2.1.15. *In the special case where all elements of* relbd(B) *are ground points, the five directions* $\mathbf{p}-\mathbf{q}$, $\frac{1}{2}\,(\mathbf{q}+\mathbf{a})-\mathbf{b}$, $\frac{1}{2}\,(\mathbf{q}+\mathbf{b})-\mathbf{a}$, $\mathbf{e}_3$ *and* $-\mathbf{e}_3$ *illuminate* K.

Proof. In this special case, $W=\mathrm{Pr}^{-1}\big(\mathrm{relbd}(B)\big)\cap\mathrm{bd}(K)=\mathrm{relbd}(B)$ as a result of every element from relbd(B) being a ground point. Recall from Proposition 4.2.2.1.11 that $\mathrm{relbd}(B)=\big(B_{\mathbf{p}}\cap\mathrm{relbd}(B)\big)\cup\big(B_{\mathbf{q}}\cap\mathrm{relbd}(B)\big)$. It follows from Proposition 4.2.2.1.10 that all elements of $B_{\mathbf{q}}\cap\mathrm{relbd}(B)$ are illuminated by $\mathbf{p}-\mathbf{q}$ and it follows from Case 1 of Lemma 4.2.2.1.12 that any element of $B_{\mathbf{p}}\cap\mathrm{relbd}(B)$ is illuminated by either $\frac{1}{2}\,(\mathbf{q}+\mathbf{a})-\mathbf{b}$ or $\frac{1}{2}\,(\mathbf{q}+\mathbf{b})-\mathbf{a}$. Thus, W is illuminated by the three directions $\mathbf{p}-\mathbf{q}$, $\frac{1}{2}\,(\mathbf{q}+\mathbf{a})-\mathbf{b}$ and $\frac{1}{2}\,(\mathbf{q}+\mathbf{b})-\mathbf{a}$.

To see that the direction $-\mathbf{e}_3$ will illuminate $\mathrm{relint}(B)_{+}$, observe the following argument.

Suppose $\mathbf{x} \in \text{relint}(B)_{+} = \text{Pr}^{-1}(\text{relint}(B)) \cap \text{bd}(K) \cap H_{+}$. Notice that this means $\mathbf{x} \in \text{Pr}^{-1}(\text{relint}(B)) = \{\mathbf{z} \in \mathbb{E}^3 \mid \text{Pr}(\mathbf{z}) \in \text{relint}(B)\}$. This implies that $\text{Pr}(\mathbf{x}) \in \text{relint}(B)$. Just like in Case 2 of Lemma 4.2.2.1.12, there exists some real number $0 < \rho \leqslant 1$ such that $\mathbf{x} = \text{Pr}(\mathbf{x}) + \rho \cdot \frac{\mathcal{T}}{2}\mathbf{e}_3$. It follows that

$$\mathbf{x} + \rho \cdot \frac{\mathcal{T}}{2}(-\mathbf{e}_3) = \text{Pr}(\mathbf{x}) + \rho \cdot \frac{\mathcal{T}}{2}\mathbf{e}_3 - \rho \cdot \frac{\mathcal{T}}{2}\mathbf{e}_3 = \text{Pr}(\mathbf{x}) \in \text{relint}(B),$$

which shows that the vector $-\mathbf{e}_3$ illuminates any element in $\text{relint}(B)_{+}$.

A similar argument can be used to show that the direction $\mathbf{e}_3$ illuminates $\text{relint}(B)_{-}$.

Recall from Proposition 4.1.2 that $\text{bd}(K) = W \cup \text{relint}(B)_{+} \cup \text{relint}(B)_{-}$ and that the sets W, $\text{relint}(B)_{+}$ and $\text{relint}(B)_{-}$ are pairwise disjoint. Hence, the directions $\mathbf{p}-\mathbf{q}$, $\frac{1}{2}(\mathbf{q}+\mathbf{a})-\mathbf{b}$, $\frac{1}{2}(\mathbf{q}+\mathbf{b})-\mathbf{a}$, $\mathbf{e}_3$ and $-\mathbf{e}_3$ illuminate K, in this special case. ∎

4.2.2.2 Suppose that q is a cliff point.

In general, the direction $\mathbf{p}-\mathbf{q}$ does not illuminate the cliff, $[\mathbf{q}^{-}, \mathbf{q}^{+}]$, at $\mathbf{q}$.

Proposition 4.2.2.2.1. *Let $\tau = \|\mathbf{q}^{+} - \mathbf{q}^{-}\|$. The directions $\mathbf{p}-\mathbf{q}-\tau\mathbf{e}_3$ and $\mathbf{p}-\mathbf{q}+\tau\mathbf{e}_3$ illuminate the cliff $[\mathbf{q}^{-}, \mathbf{q}^{+}]$ at $\mathbf{q}$.*

Proof. Let $\mathbf{z} \in [\mathbf{q}, \mathbf{q}^{+}]$ be arbitrary. This means that there exists $0 \leqslant \varpi \leqslant 1$ such that $\mathbf{z} = (1-\varpi)\mathbf{q} + \varpi\mathbf{q}^{+}$. Recall from Claim (i) in §4.2.2.1 that $(\mathbf{p}, \mathbf{q}) \subseteq \text{int}(K)$; therefore, ${}^{1}\!/_{2}(\mathbf{p}+\mathbf{q}) \in \text{int}(K)$. It follows from Theorem 2.10.10 that $[{}^{1}\!/_{2}(\mathbf{p}+\mathbf{q}), \mathbf{q}^{+}) \subseteq \text{int}(K)$ and $(\mathbf{q}^{-}, {}^{1}\!/_{2}(\mathbf{p}+\mathbf{q})] \subseteq \text{int}(K)$. Therefore, ${}^{1}\!/_{2}\big({}^{1}\!/_{2}(\mathbf{p}+\mathbf{q}) + \mathbf{q}^{+}\big), {}^{1}\!/_{2}\big({}^{1}\!/_{2}(\mathbf{p}+\mathbf{q}) + \mathbf{q}^{-}\big) \in \text{int}(K)$. By Theorem 2.10.15,

$$\left[\frac{1}{2}(\mathbf{p}+\mathbf{q}), \frac{1}{2}\left(\frac{1}{2}(\mathbf{p}+\mathbf{q}) + \mathbf{q}^{+}\right)\right] \subseteq \text{int}(K)$$

and

$$\left[\frac{1}{2}\left(\frac{1}{2}(\mathbf{p}+\mathbf{q}) + \mathbf{q}^{-}\right), \frac{1}{2}(\mathbf{p}+\mathbf{q})\right] \subseteq \text{int}(K).$$

The ray $r^{\mathbf{z}}_{\mathbf{p}-\mathbf{q}-\tau\mathbf{e}_3}$ intersects the closed line segment $\big[{}^{1}\!/_{2}({}^{1}\!/_{2}(\mathbf{p}+\mathbf{q}) + \mathbf{q}^{-}), {}^{1}\!/_{2}(\mathbf{p}+\mathbf{q})\big]$. To see this, observe that the element $\mathbf{z} + \frac{1+\varpi}{4}(\mathbf{p}-\mathbf{q}-\tau\mathbf{e}_3)$ of the ray $r^{\mathbf{z}}_{\mathbf{p}-\mathbf{q}-\tau\mathbf{e}_3}$ can be re-written as follows:

$$\begin{aligned}
\mathbf{z} + \frac{1+\varpi}{4}(\mathbf{p}-\mathbf{q}-\tau\mathbf{e}_3) &= (1-\varpi)\mathbf{q} + \varpi\mathbf{q}^+ + \frac{1+\varpi}{4}(\mathbf{p}-\mathbf{q}-\tau\mathbf{e}_3) \\
&= (1-\varpi)\mathbf{q} + \varpi\left(\mathbf{q}+\frac{\tau}{2}\mathbf{e}_3\right) + \frac{1+\varpi}{4}(\mathbf{p}-\mathbf{q}-\tau\mathbf{e}_3) \\
&= \frac{1-\varpi}{2}\left(\mathbf{q}-\frac{\tau}{2}\mathbf{e}_3\right) + \frac{\varpi}{2}\mathbf{q} + \frac{\varpi+\varpi-\varpi}{4}\mathbf{p} - \frac{\varpi}{4}\mathbf{q} + \frac{1}{4}(\mathbf{p}+\mathbf{q}) \\
&= \frac{1-\varpi}{2}\left(\frac{1}{2}(\mathbf{p}+\mathbf{q})+\mathbf{q}^-\right) + \frac{\varpi}{2}(\mathbf{p}+\mathbf{q}) \\
&\subseteq \left[\frac{1}{2}\left(\frac{1}{2}(\mathbf{p}+\mathbf{q})+\mathbf{q}^-\right), \frac{1}{2}(\mathbf{p}+\mathbf{q})\right] \subseteq \mathrm{int}(K),
\end{aligned}$$

for $0 < \frac{1}{4} \leqslant \dfrac{\varpi}{2} \leqslant \frac{1}{2} < 1$.

Let $\mathbf{z}' \in [\mathbf{q}^-, \mathbf{q}]$ be arbitrary. A nearly identical proof to the one directly above will show that the ray $r^{\mathbf{z}'}_{\mathbf{p}-\mathbf{q}+\tau\mathbf{e}_3}$ intersects the line segment $\left[{}^1\!/_2\,(\mathbf{p}+\mathbf{q})\,, {}^1\!/_2\,({}^1\!/_2\,(\mathbf{p}+\mathbf{q})+\mathbf{q}^+)\right]$ and therefore, intersects $\mathrm{int}(K)$. ■

Proposition 4.2.2.2.2. *There exists a real number $\chi' > 0$ such that the directions $\mathbf{p}-\mathbf{q}+\tau\mathbf{e}_3$ and $\mathbf{p}-\mathbf{q}-\tau\mathbf{e}_3$ illuminate, $\left(W_{\ell'\cap\mathrm{relbd}(B)} + \chi'\mathrm{B}(\mathbf{o},1)\right) \cap \mathrm{bd}(K)$, an open neighbourhood of $W_{\ell'\cap\mathrm{relbd}(B)}$ on $\mathrm{bd}(K)$.*

Proof. Let $I = [{}^1\!/_2\,({}^1\!/_2\,(\mathbf{p}+\mathbf{q})+\mathbf{q}^-)\,, {}^1\!/_2\,(\mathbf{p}+\mathbf{q})] \cup [{}^1\!/_2\,(\mathbf{p}+\mathbf{q})\,, {}^1\!/_2\,({}^1\!/_2\,(\mathbf{p}+\mathbf{q})+\mathbf{q}^+)]$ and let $\chi' = \inf\{\|\mathbf{x}_i - \mathbf{x}_k\| \mid \mathbf{x}_i \in I,\ \mathbf{x}_k \in \mathrm{bd}(K)\}$. It is important to verify that $\chi' > 0$ and $\mathrm{B}(\mathbf{n}, \chi') \subseteq K$, for any $\mathbf{n} \in I$.

First, observe that Proposition 2.10.5 and Corollary 2.10.8 imply that

$$\left[{}^1\!/_2\left({}^1\!/_2\,(\mathbf{p}+\mathbf{q})+\mathbf{q}^-\right), {}^1\!/_2\,(\mathbf{p}+\mathbf{q})\right] \qquad \text{and} \qquad \left[{}^1\!/_2\,(\mathbf{p}+\mathbf{q})\,, {}^1\!/_2\left({}^1\!/_2\,(\mathbf{p}+\mathbf{q})+\mathbf{q}^+\right)\right]$$

are compact. It follows by Proposition 2.8.3 that

$$\left[\frac{1}{2}\left(\frac{1}{2}(\mathbf{p}+\mathbf{q})+\mathbf{q}^-\right), \frac{1}{2}(\mathbf{p}+\mathbf{q})\right] \bigcup \left[\frac{1}{2}(\mathbf{p}+\mathbf{q}), \frac{1}{2}\left(\frac{1}{2}(\mathbf{p}+\mathbf{q})+\mathbf{q}^+\right)\right]$$

is compact. Furthermore, note that $\mathrm{bd}(K)$ is closed. This means that by Theorem 2.8.5, there exist elements $\mathbf{k}_1 \in I$ and $\mathbf{k}_2 \in \mathrm{bd}(K)$ such that

$$\|\mathbf{k}_1 - \mathbf{k}_2\| = \inf\{\|\mathbf{x}_i - \mathbf{x}_k\| \mid \mathbf{x}_i \in I,\ \mathbf{x}_k \in \mathrm{bd}(K)\}$$

$$= \chi'.$$

Recall from Proposition 4.2.2.2.1 that

$$\left[\frac{1}{2}\left(\frac{1}{2}(\mathbf{p}+\mathbf{q})+\mathbf{q}^-\right), \frac{1}{2}(\mathbf{p}+\mathbf{q})\right], \left[\frac{1}{2}(\mathbf{p}+\mathbf{q}), \frac{1}{2}\left(\frac{1}{2}(\mathbf{p}+\mathbf{q})+\mathbf{q}^+\right)\right] \subseteq \mathrm{int}(K).$$

This means that

$$\left[\frac{1}{2}\left(\frac{1}{2}(\mathbf{p}+\mathbf{q})+\mathbf{q}^-\right), \frac{1}{2}(\mathbf{p}+\mathbf{q})\right] \bigcup \left[\frac{1}{2}(\mathbf{p}+\mathbf{q}), \frac{1}{2}\left(\frac{1}{2}(\mathbf{p}+\mathbf{q})+\mathbf{q}^+\right)\right] \subseteq \mathrm{int}(K).$$

This together with Theorem 2.5.7 implies that

$$\left(\left[\frac{1}{2}\left(\frac{1}{2}(\mathbf{p}+\mathbf{q})+\mathbf{q}^-\right), \frac{1}{2}(\mathbf{p}+\mathbf{q})\right] \bigcup \left[\frac{1}{2}(\mathbf{p}+\mathbf{q}), \frac{1}{2}\left(\frac{1}{2}(\mathbf{p}+\mathbf{q})+\mathbf{q}^+\right)\right]\right) \bigcap \mathrm{bd}(K) = \varnothing.$$

Hence, $\mathbf{k}_1 \neq \mathbf{k}_2$ and therefore, $\chi' > 0$.

Finally, let $\mathbf{n} \in I$ and $\mathbf{g} \in \mathrm{B}(\mathbf{n}, \chi')$ be arbitrarily chosen. Then,

$$\|\mathbf{g}-\mathbf{n}\| < \chi' \leqslant \inf\{\|\mathbf{n}-\mathbf{x}_k\| \mid \mathbf{x}_k \in \mathrm{bd}(K)\} \leqslant \|\mathbf{k}-\mathbf{n}\|,$$

for all $\mathbf{k} \in \mathrm{bd}(K)$. This means $\mathbf{g} \notin \mathrm{bd}(K)$.

Since $\mathbf{n} \in \mathrm{int}(K)$, it follows by Corollary 2.10.12 that there exists $\widehat{\mathbf{k}} \in \mathrm{bd}(K)$ such that $\left\{\widehat{\mathbf{k}}\right\} = r^{\mathbf{n}}_{\mathbf{g}-\mathbf{n}} \cap \mathrm{bd}(K)$. Since K is convex, $\left[\mathbf{n}, \widehat{\mathbf{k}}\right] \subseteq K$. Moreover,

$$\mathbf{g} = \left(1 - \frac{\|\mathbf{g}-\mathbf{n}\|}{\left\|\widehat{\mathbf{k}}-\mathbf{n}\right\|}\right)\mathbf{n} + \frac{\|\mathbf{g}-\mathbf{n}\|}{\left\|\widehat{\mathbf{k}}-\mathbf{n}\right\|}\widehat{\mathbf{k}} \in \left[\mathbf{n}, \widehat{\mathbf{k}}\right],$$

since $\|\mathbf{g}-\mathbf{n}\| < \left\|\widehat{\mathbf{k}}-\mathbf{n}\right\|$.

Hence, $\mathrm{B}(\mathbf{n}, \chi') \subseteq K$, for any $\mathbf{n} \in I$.

It is, also, important to verify that $W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o}, 1) \cap \mathrm{bd}(K)$ contains $W_{\ell' \cap \mathrm{relbd}(B)}$ and is open in $\mathrm{bd}(K)$.

Let $\mathbf{x} \in W_{\ell' \cap \mathrm{relbd}(B)}$ be arbitrarily chosen. Note that

$$\chi' \mathrm{B}(\mathbf{o}, 1) = \{\chi' \mathbf{z} \mid \|\mathbf{z}\| < 1\}$$

$$= \left\{ \mathbf{z} \in \mathbb{E}^3 \;\middle|\; \frac{1}{\chi'} \|\mathbf{z}\| < 1 \right\}$$

$$= \left\{ \mathbf{z} \in \mathbb{E}^3 \mid \|\mathbf{z}\| < \chi' \right\} = \mathrm{B}\left(\mathbf{o}, \chi'\right). \tag{4.32}$$

Clearly, $\mathbf{o} \in \chi' \mathrm{B}(\mathbf{o}, 1)$. It follows that $\mathbf{x} = \mathbf{x} + \mathbf{o} \in W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o}, 1)$. Thus,

$$W_{\ell' \cap \mathrm{relbd}(B)} \subseteq W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o}, 1).$$

Recall that $W_{\ell' \cap \mathrm{relbd}(B)} = \mathrm{Pr}^{-1}(\ell' \cap \mathrm{relbd}(B)) \cap \mathrm{bd}(K)$. This means

$$W_{\ell' \cap \mathrm{relbd}(B)} = W_{\ell' \cap \mathrm{relbd}(B)} \cap \mathrm{bd}(K) \subseteq \left(W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o}, 1)\right) \cap \mathrm{bd}(K).$$

It follows, by Equation 4.32, that $W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o}, 1) = W_{\ell' \cap \mathrm{relbd}(B)} + \mathrm{B}(\mathbf{o}, \chi')$. By expanding and simplifying,

$$W_{\ell' \cap \mathrm{relbd}(B)} + \mathrm{B}(\mathbf{o}, \chi') = \bigcup_{\mathbf{x} \in W_{\ell' \cap \mathrm{relbd}(B)}} (\mathbf{x} + \mathrm{B}(\mathbf{o}, \chi')) = \bigcup_{\mathbf{x} \in W_{\ell' \cap \mathrm{relbd}(B)}} \mathrm{B}(\mathbf{x}, \chi').$$

By ***(i)*** and ***(iv)*** of Theorem 2.5.1, $\bigcup_{\mathbf{x} \in W_{\ell' \cap \mathrm{relbd}(B)}} \mathrm{B}(\mathbf{x}, \chi')$ is open.

Thus, $W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o}, 1)$ is open in $\mathbb{E}^3$. Equipping $\mathrm{bd}(K)$ with the subspace topology $\mathcal{T}_{\mathrm{bd}(K)} = \{V \cap \mathrm{bd}(K) \mid V \text{ is open in } \mathbb{E}^3\}$, it can be seen that

$$\left(W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o}, 1)\right) \cap \mathrm{bd}(K) \in \mathcal{T}_{\mathrm{bd}(K)}$$

and therefore, $\left(W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o}, 1)\right) \cap \mathrm{bd}(K)$ is open in $\mathrm{bd}(K)$.

Finally, it must be verified that $\left(W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o}, 1)\right) \cap \mathrm{bd}(K)$ is illuminated by either $\mathbf{p} - \mathbf{q} + \tau \mathbf{e}_3$ or $\mathbf{p} - \mathbf{q} - \tau \mathbf{e}_3$.

Let $\mathbf{z} \in \left(W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o}, 1)\right) \cap \mathrm{bd}(K)$ be arbitrarily chosen. It follows, by definition, that there exists a scalar $0 \leqslant \mu' < 1$ and a unit vector $\mathbf{v}$ such that

$$\mathbf{z} = \mathbf{x} + \mu' \chi' \mathbf{v},$$

for some $\mathbf{x} \in W_{\ell' \cap \mathrm{relbd}(B)}$. Here, in §4.2.2.2,

$$W_{\ell' \cap \mathrm{relbd}(B)} = \mathrm{Pr}^{-1}(\ell' \cap \mathrm{relbd}(B)) \cap \mathrm{bd}(K) = \mathrm{Pr}^{-1}(\{\mathbf{q}\}) \cap \mathrm{bd}(K) = [\mathbf{q}^-, \mathbf{q}^+].$$

It follows from the previous Proposition 4.2.2.2.1 that $\mathbf{x}$ can be illuminated by either $\mathbf{p}-\mathbf{q}+\tau\mathbf{e}_3$ or $\mathbf{p}-\mathbf{q}-\tau\mathbf{e}_3$. Suppose, without loss of generality, that $\mathbf{x}$ is illuminated by $\mathbf{p}-\mathbf{q}-\tau\mathbf{e}_3$. This means that there exists a real number $\lambda' > 0$ such that

$$\mathbf{x} + \lambda' (\mathbf{p} - \mathbf{q} - \tau\mathbf{e}_3) \in I \subseteq \operatorname{int}(K).$$

It follows from above that

$$\mathrm{B}\left(\mathbf{x} + \lambda' (\mathbf{p} - \mathbf{q} - \tau\mathbf{e}_3), \chi'\right) \subseteq K.$$

Let $\mathbf{z}'$ be an arbitrarily chosen element from $\mathrm{B}\Big(\mathbf{z} + \lambda (\mathbf{p} - \mathbf{q} - \tau\mathbf{e}_3), \chi' (1 - \mu')\Big)$, where the point $\mathbf{z} + \lambda (\mathbf{p} - \mathbf{q} - \tau\mathbf{e}_3)$ is an element of the ray $r^{\mathbf{z}}_{\mathbf{p}-\mathbf{q}-\tau\mathbf{e}_3}$. Observe that

$$\begin{aligned}
&\|\mathbf{z}' - (\mathbf{x} + \lambda' (\mathbf{p} - \mathbf{q} - \tau\mathbf{e}_3))\| \\
&\leqslant \|\mathbf{z}' - (\mathbf{z} + \lambda' (\mathbf{p} - \mathbf{q} - \tau\mathbf{e}_3))\| + \|\mathbf{z} + \lambda' (\mathbf{p} - \mathbf{q} - \tau\mathbf{e}_3) - (\mathbf{x} + \lambda' (\mathbf{p} - \mathbf{q} - \tau\mathbf{e}_3))\| \\
&< \chi' (1 - \mu') + \|\mathbf{x} + \mu'\chi'\mathbf{v} + \lambda' (\mathbf{p} - \mathbf{q} - \tau\mathbf{e}_3) - (\mathbf{x} + \lambda' (\mathbf{p} - \mathbf{q} - \tau\mathbf{e}_3))\| \\
&= \chi' (1 - \mu') + \mu'\chi'\|\mathbf{v}\| = \chi'.
\end{aligned}$$

This implies that

$$\mathrm{B}\left(\mathbf{z} + \lambda' (\mathbf{p} - \mathbf{q} - \tau\mathbf{e}_3), \chi' (1 - \mu')\right) \subseteq \mathrm{B}\left(\mathbf{x} + \lambda' (\mathbf{p} - \mathbf{q} - \tau\mathbf{e}_3), \chi'\right).$$

Thus, $\mathbf{z} + \lambda' (\mathbf{p} - \mathbf{q} - \tau\mathbf{e}_3) \in r^{\mathbf{z}}_{\mathbf{p}-\mathbf{q}-\tau\mathbf{e}_3} \cap \operatorname{int}(K)$. ■

It follows from Lemma 4.2.2.1.3, Proposition 4.2.2.1.8 and Proposition 4.2.2.1.9 that there exists points $\mathbf{a}, \mathbf{b} \in \operatorname{relbd}(B) \cap \big(W_{\ell' \cap \operatorname{relbd}(B)} + \chi'\mathrm{B}(\mathbf{o}, 1)\big)$ such that the points $\mathbf{p}, \mathbf{a}, \mathbf{q}$ and $\mathbf{b}$ follow each other in this order when travelling counter-clockwise on $\operatorname{relbd}(B)$ starting at $\mathbf{p}$. Moreover, the line passing through $\mathbf{a}$ and $\mathbf{b}$ is parallel to ℓ and ℓ'. By Lemma 4.2.2.1.12, the directions $\frac{1}{2}(\mathbf{q}+\mathbf{a}) - \mathbf{b}$, $\frac{1}{2}(\mathbf{q}+\mathbf{b}) - \mathbf{a}$, $\left(1 - \frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{a}) - \mathbf{b}\right) \pm \mathcal{T}\mathbf{e}_3$ and $\left(1 - \frac{2(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{b}) - \mathbf{a}\right) \pm \mathcal{T}\mathbf{e}_3$ will illuminate $\operatorname{bd}(K) \backslash \big(W_{\ell' \cap \operatorname{relbd}(B)} + \chi'\mathrm{B}(\mathbf{o}, 1)\big)$.

This means that the eight directions $\mathbf{p} - \mathbf{q} \pm \tau\mathbf{e}_3$ $\frac{1}{2}(\mathbf{q}+\mathbf{a}) - \mathbf{b}$, $\frac{1}{2}(\mathbf{q}+\mathbf{b}) - \mathbf{a}$, $\left(1 - \frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{a}) - \mathbf{b}\right) \pm \mathcal{T}\mathbf{e}_3$ and $\left(1 - \frac{2(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{b}) - \mathbf{a}\right) \pm \mathcal{T}\mathbf{e}_3$ illuminate $\operatorname{bd}(K)$.

4.2.3 Second Major Case of Theorem 4.1

In this second case, the illumination of K is constructed when $\text{relbd}(B)$ contains at least one side and has the property that either:

(i) the complete antipode of the midpoint of one of the sides of $\text{relbd}(B)$ is a single point; or

(ii) for each side of $\text{relbd}(B)$, there exists another side parallel to it and the wall through one side of $\text{relbd}(B)$ is degenerate.

4.2.3.1 Suppose $\text{relbd}(B)$ contains a side such that the complete antipode of its midpoint $\mathbf{p}$ is a single point $\mathbf{q}$.

If $\mathbf{q}$ is a ground point, then K can be illuminated using the exact same procedure as in §4.2.2.1. This means 7 directions illuminate K.

If $\mathbf{q}$ is a cliff point, then K can be illuminated using the exact same procedure as in §4.2.2.2. This means 8 directions illuminate K.

4.2.3.2 For each side of $\text{relbd}(B)$, suppose that there exists another side parallel to it. Furthermore, suppose that at least one wall through a side of $\text{relbd}(B)$ is degenerate.

Let $[\mathbf{u}, \mathbf{v}] \subseteq \text{relbd}(B)$ be a side with endpoints $\mathbf{u}, \mathbf{v}$ such that $W_{[\mathbf{u},\mathbf{v}]} = [\mathbf{u}, \mathbf{v}]$. Denote the other side of B parallel to $[\mathbf{u}, \mathbf{v}]$ by $[\mathbf{w}, \mathbf{z}]$ where the points $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and $\mathbf{z}$ follow each other in this order when travelling counter-clockwise on $\text{relbd}(B)$, starting at the point $\mathbf{u}$. Since the sides $[\mathbf{u}, \mathbf{v}]$ and $[\mathbf{w}, \mathbf{z}]$ are parallel, there exists a real number $\hbar > 0$ such that $\mathbf{w} - \mathbf{z} = \hbar\,(\mathbf{v} - \mathbf{u})$. Let $\mathbf{p} = \frac{1}{2}(\mathbf{w} + \mathbf{z})$ and let $\mathbf{q} = \frac{1}{2}(\mathbf{u} + \mathbf{v})$. Moreover, let ℓ denote the supporting line of B at the side $[\mathbf{z}, \mathbf{w}]$ and let ℓ' be the supporting line of B at the side $[\mathbf{u}, \mathbf{v}]$. It follows that ℓ and ℓ' are parallel and therefore, there exists a vector $\mathbf{t} \neq \mathbf{o}$ such that $\ell' = \ell + \mathbf{t}$.

Proposition 4.2.3.2.1. *The directions* $\mathbf{p} - \mathbf{u}$ *and* $\mathbf{p} - \mathbf{v}$ *will illuminate the side* $[\mathbf{u}, \mathbf{v}]$*.*

Proof. As established in Properties 4.1.1, B is convex. It follows that $[\mathbf{p}, \mathbf{u}], [\mathbf{p}, \mathbf{v}] \subseteq B$. Suppose for a contradiction that $[\mathbf{p}, \mathbf{u}] \subseteq \mathrm{relbd}(B)$. Then, there exists a supporting line of B which contains the closed line segment $[\mathbf{p}, \mathbf{u}]$; denote it by ℓ_1. Recall that $\mathbf{p}$ is the midpoint of the side $[\mathbf{w}, \mathbf{z}]$. Therefore, the lines ℓ and ℓ_1 support B at $\mathbf{p}$. It follows from Theorem 2.2.2.1 that ℓ_1 is not parallel to ℓ or ℓ' since it intersects these lines at the points $\mathbf{p}$ and $\mathbf{u}$, respectively. Notice that $\ell_1 = \{\mathbf{p} + \lambda(\mathbf{u} - \mathbf{p}) \mid \lambda \in \mathbb{R}\}$. It follows from Theorem 2.10.1.3 that B should be completely contained in one of the closed half-spaces determined by ℓ_1. However, notice that the points $\mathbf{p}$, $\mathbf{u}$, $\mathbf{w} = \mathbf{p} + \frac{\hbar}{2}(\mathbf{v} - \mathbf{u})$, $\mathbf{q} = \mathbf{u} + \frac{1}{2}(\mathbf{v} - \mathbf{u})$, and $\mathbf{v} = \mathbf{u} + (\mathbf{v} - \mathbf{u})$ lie in the closed half-space $\{\mathbf{p} + \lambda(\mathbf{u} - \mathbf{p}) + \mu(\mathbf{v} - \mathbf{u}) \mid \lambda, \mu \in \mathbb{R} \text{ such that } \mu \geqslant 0\}$ determined by ℓ_1, but the point $\mathbf{z} = \mathbf{p} + \frac{\hbar}{2}(\mathbf{u} - \mathbf{v}) \in B$ does not lie in the same closed half-space determined by ℓ_1. This is a contradiction. Therefore, $(\mathbf{p}, \mathbf{u}) \nsubseteq \mathrm{relbd}(B)$ since $\mathbf{p}, \mathbf{u} \in \mathrm{relbd}(B)$. Recall from Properties 4.1.1 that B is closed and has non-empty interior in the x_1x_2-plane. This means that $B = \mathrm{relint}(B) \cup \mathrm{relbd}(B)$. Also, recall that $\mathrm{relint}(B) \cap \mathrm{relbd}(B) = \varnothing$. Thus, $(\mathbf{p}, \mathbf{u}) \subseteq \mathrm{relint}(B) \subseteq \mathrm{int}(K)$. A nearly identical argument can be used to show that $(\mathbf{p}, \mathbf{v}) \subseteq \mathrm{relint}(B) \subseteq \mathrm{int}(K)$.

This implies that the rays $r^{\mathbf{u}}_{\mathbf{p}-\mathbf{u}}$ and $r^{\mathbf{v}}_{\mathbf{p}-\mathbf{v}}$ intersect $\mathrm{relint}(B) \subseteq \mathrm{int}(K)$. In other words, the direction $\mathbf{p} - \mathbf{u}$ illuminates the point $\mathbf{u}$ and the direction $\mathbf{p} - \mathbf{v}$ illuminates the point $\mathbf{v}$.

Let $\mathbf{x} \in (\mathbf{u}, \mathbf{v})$ be arbitrarily chosen. This means that there exists $0 < \eth < 1$ such that $\mathbf{x} = \eth\mathbf{u} + (1 - \eth)\mathbf{v}$. Consider the point $\mathbf{x} + \eth(\mathbf{p} - \mathbf{u})$ on the ray $r^{\mathbf{x}}_{\mathbf{p}-\mathbf{u}}$:

$$\begin{aligned} \mathbf{x} + \eth(\mathbf{p} - \mathbf{u}) &= \eth\mathbf{u} + (1 - \eth)\mathbf{v} + \eth(\mathbf{p} - \mathbf{u}) \\ &= \eth\mathbf{p} + (1 - \eth)\mathbf{v} \in (\mathbf{p}, \mathbf{v}) \subseteq \mathrm{relint}(B). \end{aligned}$$

This means that direction $\mathbf{p} - \mathbf{u}$ illuminates all points in the open line segment $(\mathbf{u}, \mathbf{v})$. ■

A similar argument to the one used in the proof of Proposition 4.2.2.2.2 can be used to show that there exists a real number $\chi' > 0$ such that the directions $\mathbf{p} - \mathbf{u}$ and $\mathbf{p} - \mathbf{v}$

illuminate the set $W_{\ell' \cap \mathrm{relbd}(B)} + \mathrm{B}\left(\mathbf{o}, \chi\right)$, which is an open neighbourhood around the side $[\mathbf{u}, \mathbf{v}]$ on the $\mathrm{bd}(K)$.

Then, by Lemma 4.2.2.1.3, Proposition 4.2.2.1.8 and Proposition 4.2.2.1.9, there exists points $\mathbf{a}, \mathbf{b} \in \mathrm{relbd}(B) \cap \left(W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}\left(\mathbf{o}, 1\right)\right)$ such that the points $\mathbf{p}, \mathbf{z}, \mathbf{a}, \mathbf{u}, \mathbf{q}, \mathbf{v}, \mathbf{b}$ and $\mathbf{w}$ follow each other in this order when travelling counter-clockwise on $\mathrm{relbd}(B)$ starting at $\mathbf{p}$. Moreover, the line passing through $\mathbf{a}$ and $\mathbf{b}$ is parallel to ℓ and ℓ'. By Lemma 4.2.2.1.12, the directions $\frac{1}{2}\left(\mathbf{q} + \mathbf{a}\right) - \mathbf{b}$, $\frac{1}{2}\left(\mathbf{q} + \mathbf{b}\right) - \mathbf{a}$, $\left(1 - \frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}\left(\mathbf{q} + \mathbf{a}\right) - \mathbf{b}\right) \pm \mathcal{T}\mathbf{e}_3$ and $\left(1 - \frac{2(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}\left(\mathbf{q} + \mathbf{b}\right) - \mathbf{a}\right) \pm \mathcal{T}\mathbf{e}_3$ will illuminate $\mathrm{bd}(K) \backslash \left(W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}\left(\mathbf{o}, 1\right)\right)$.

This means that the eight directions $\mathbf{p} - \mathbf{u}$, $\mathbf{p} - \mathbf{v}$, $\frac{1}{2}\left(\mathbf{q} + \mathbf{a}\right) - \mathbf{b}$, $\frac{1}{2}\left(\mathbf{q} + \mathbf{b}\right) - \mathbf{a}$, $\left(1 - \frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}\left(\mathbf{q} + \mathbf{a}\right) - \mathbf{b}\right) \pm \mathcal{T}\mathbf{e}_3$ and $\left(1 - \frac{2(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}\left(\mathbf{q} + \mathbf{b}\right) - \mathbf{a}\right) \pm \mathcal{T}\mathbf{e}_3$ illuminate $\mathrm{bd}(K)$.

4.2.4 Third Major Case of Theorem 4.1

Recall from Properties 4.1.1 that B is a convex body in the x_1x_2-plane. It follows that B is a convex body in $\mathbb{E}^2$. By the John - Löwner Theorem in $\mathbb{E}^2$, there exists a unique ellipse $\mathcal{E}$ of maximal volume such that $\mathcal{E} \subset B \subset 2\,\mathcal{E}$. It follows, by definition, that there exists an invertible linear transformation $T : \mathbb{E}^2 \to \mathbb{E}^2$ and a vector $\mathbf{a} \in \mathbb{E}^2$ such that

$$\mathcal{E} = T\left(\mathrm{B}^2[\mathbf{o}, 1]\right) + \mathbf{a}.$$

Let $T' : \mathbb{E}^3 \to \mathbb{E}^3$ be the linear transformation induced by the 3×3 block matrix

$$\begin{bmatrix} T & 0 \\ 0 & 1 \end{bmatrix}$$

and let $\mathbf{a}' = \langle \mathbf{a}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{a}, \mathbf{e}_2 \rangle \mathbf{e}_2$ where $\mathbf{e}_1$ and $\mathbf{e}_2$ are two of the standard basis vectors of $\mathbb{E}^3$. Then, in the x_1x_2-plane of $\mathbb{E}^3$,

$$\mathcal{E} = T'\Big(\mathrm{B}^3[\mathbf{o}, 1] \cap \left(\mathbb{E}^2 \times \{0\}\right)\Big) + \mathbf{a}'.$$

It follows from *(ii)* of Properties 3.3.2 that $\det(T) \neq 0$ since T is invertible. This together with *(iv)* of Properties 3.3.2 imply that $\det(T') = \det(T)\det(1) = \det(T) = \neq 0$. Therefore, the linear transformation T' is also invertible. It follows from *(xi)* of Properties 2.3.1 that

$$(T')^{-1} = \begin{bmatrix} T^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

Apply the linear transformation

$$T^{\star} = \begin{bmatrix} \frac{1}{2}T^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

to K.

It will be shown that the convex body $T^{\star}(K)$ is affine plane symmetric about the x_1x_2-plane. Let $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{E}^2$ denote the rows of the 2×2 matrix T^{-1}. Then, for some arbitrarily chosen $\mathbf{k} \in K$,

$$T^{\star}(\mathbf{k}) = \left\langle \frac{1}{2}\begin{bmatrix} \mathbf{t}_1 \\ 0 \end{bmatrix}, \mathbf{k} \right\rangle \mathbf{e}_1 + \left\langle \frac{1}{2}\begin{bmatrix} \mathbf{t}_2 \\ 0 \end{bmatrix}, \mathbf{k} \right\rangle \mathbf{e}_2 + \left\langle \begin{bmatrix} \mathbf{o} \\ 1 \end{bmatrix}, \mathbf{k} \right\rangle \mathbf{e}_3.$$

Since K is affine plane symmetric about the x_1x_2-plane, by assumption, it follows that there exists $\mathbf{k}' \in K$ such that $\frac{1}{2}(\mathbf{k} + \mathbf{k}') \in B$ and $\mathbf{k}' \in \{\mathbf{k} + \lambda \mathbf{e}_3 \mid \lambda \in \mathbb{R}\}$. Therefore,

$$\begin{aligned}
\frac{1}{2}\Big(T^{\star}(\mathbf{k}) + T^{\star}(\mathbf{k}')\Big) &= \frac{1}{2}\Bigg(\left\langle \frac{1}{2}\begin{bmatrix} \mathbf{t}_1 \\ 0 \end{bmatrix}, \mathbf{k} \right\rangle \mathbf{e}_1 + \left\langle \frac{1}{2}\begin{bmatrix} \mathbf{t}_2 \\ 0 \end{bmatrix}, \mathbf{k} \right\rangle \mathbf{e}_2 + \left\langle \begin{bmatrix} \mathbf{o} \\ 1 \end{bmatrix}, \mathbf{k} \right\rangle \mathbf{e}_3 \\
&\quad + \left\langle \frac{1}{2}\begin{bmatrix} \mathbf{t}_1 \\ 0 \end{bmatrix}, \mathbf{k}' \right\rangle \mathbf{e}_1 + \left\langle \frac{1}{2}\begin{bmatrix} \mathbf{t}_2 \\ 0 \end{bmatrix}, \mathbf{k}' \right\rangle \mathbf{e}_2 + \left\langle \begin{bmatrix} \mathbf{o} \\ 1 \end{bmatrix}, \mathbf{k}' \right\rangle \mathbf{e}_3\Bigg) \\
&= \left\langle \frac{1}{2}\begin{bmatrix} \mathbf{t}_1 \\ 0 \end{bmatrix}, \frac{1}{2}(\mathbf{k} + \mathbf{k}') \right\rangle \mathbf{e}_1 + \left\langle \frac{1}{2}\begin{bmatrix} \mathbf{t}_2 \\ 0 \end{bmatrix}, \frac{1}{2}(\mathbf{k} + \mathbf{k}') \right\rangle \mathbf{e}_2 \\
&\quad + \left\langle \frac{1}{2}\begin{bmatrix} \mathbf{o} \\ 1 \end{bmatrix}, \frac{1}{2}(\mathbf{k} + \mathbf{k}') \right\rangle \mathbf{e}_3 \\
&= T^{\star}\left(\frac{1}{2}(\mathbf{k} + \mathbf{k}')\right) \in T^{\star}(B).
\end{aligned}$$

Moreover, there exists some $\lambda' \in \mathbb{R}$ such that

$$\begin{aligned} T^{\star}(\mathbf{k}') &= T^{\star}(\mathbf{k} + \lambda'\mathbf{e}_3) \\ &= \left\langle \frac{1}{2}\begin{bmatrix} \mathbf{t}_1 \\ 0 \end{bmatrix}, \mathbf{k} + \lambda'\mathbf{e}_3 \right\rangle \mathbf{e}_1 + \left\langle \frac{1}{2}\begin{bmatrix} \mathbf{t}_2 \\ 0 \end{bmatrix}, \mathbf{k} + \lambda'\mathbf{e}_3 \right\rangle \mathbf{e}_2 + \left\langle \begin{bmatrix} \mathbf{o} \\ 1 \end{bmatrix}, \mathbf{k} + \lambda'\mathbf{e}_3 \right\rangle \mathbf{e}_3 \\ &= T^{\star}(\mathbf{k}) + \lambda' \left\langle \frac{1}{2}\begin{bmatrix} \mathbf{t}_1 \\ 0 \end{bmatrix}, \mathbf{e}_3 \right\rangle \mathbf{e}_1 + \lambda' \left\langle \frac{1}{2}\begin{bmatrix} \mathbf{t}_2 \\ 0 \end{bmatrix}, \mathbf{e}_3 \right\rangle \mathbf{e}_2 + \lambda' \left\langle \begin{bmatrix} \mathbf{o} \\ 1 \end{bmatrix}, \mathbf{e}_3 \right\rangle \mathbf{e}_3 \\ &= T^{\star}(\mathbf{k}) + \lambda'\mathbf{e}_3 \in \left\{ T^{\star}(\mathbf{k}) + \widehat{\lambda}\mathbf{e}_3 \mid \widehat{\lambda} \in \mathbb{R} \right\}. \end{aligned}$$

Finally, notice that

$$\begin{aligned} T^{\star}(\mathcal{E}) &= T^{\star}\Big(T'\Big(\mathrm{B}^3[\mathbf{o}, 1] \cap \big(\mathbb{E}^2 \times \{0\}\big)\Big) + \mathbf{a}'\Big) \\ &= T^{\star}T'\Big(\mathrm{B}^3[\mathbf{o}, 1] \cap \big(\mathbb{E}^2 \times \{0\}\big)\Big) + T^{\star}(\mathbf{a}') \end{aligned}$$

then, by *(x)* Properties 2.3.1,

$$= \begin{bmatrix} \frac{1}{2}I_n & 0 \\ 0 & 1 \end{bmatrix} \Big(\mathrm{B}^3[\mathbf{o}, 1] \cap \big(\mathbb{E}^2 \times \{0\}\big)\Big) + T^{\star}(\mathbf{a}'),$$

where the first term,

$$\begin{bmatrix} \frac{1}{2}I_n & 0 \\ 0 & 1 \end{bmatrix} \Big(\mathrm{B}^3[\mathbf{o}, 1] \cap \big(\mathbb{E}^2 \times \{0\}\big)\Big),$$

is a disc of radius $^1/_2$ in the x_1x_2-plane. Likewise, $T^{\star}(2\mathcal{E})$ will be a disc of radius 1 in the x_1x_2-plane. Of course, $T^{\star}(\mathcal{E}) \subseteq T^{\star}(B) \subseteq T^{\star}(2\mathcal{E})$ as a result of Proposition 2.4.1.

Since the illumination number is invariant under linear transformation and $T^{\star}$ preserves the affine symmetry of K, K is hereinafter assumed to be transformed by $T^{\star}$. However, for convenience, K and all of its subsets will retain their original notation.

Accordingly, suppose that

(i) relbd(B) contains at least one side and for each side of relbd(B), there exists another side parallel to it;

(ii) each side of relbd(B) is non-degenerate;

(iii) there exists a side of relbd(B) whose length is less than $\frac{1}{2}$.

By *(iii)*, relbd(B) contains a side of length less than $\frac{1}{2}$; denote it by $[\mathbf{u}, \mathbf{v}]$. Let ℓ' denote the supporting line of B at $[\mathbf{u}, \mathbf{v}]$ and let $\mathbf{q} = \frac{1}{2}(\mathbf{u} + \mathbf{v})$. It follows from above that a disc of diameter 1 with center $\mathbf{c}$ is completely contained in B. Then, there exists a closed line segment parallel to $[\mathbf{u}, \mathbf{v}]$ in relint(B) whose midpoint is $\mathbf{c}$ and whose length is twice longer than the segment $[\mathbf{u}, \mathbf{v}]$; denote it by $[\mathbf{n}, \mathbf{m}]$ where the points $\mathbf{u}, \mathbf{v}, \mathbf{m}, \mathbf{n}$ follow each other in this order when starting at the point $\mathbf{u}$ and travelling counter-clockwise on relbd(B). It follows from *(ii)* that $[\mathbf{u}, \mathbf{v}]$ is non-degenerate and therefore, contains cliff points. Let $\mathbf{k} \in [\mathbf{u}, \mathbf{v}]$ be chosen so that

$$\left\|\mathbf{k}^{+} - \mathbf{k}^{-}\right\| = \max\left\{\left\|\mathbf{f}^{+} - \mathbf{f}^{-}\right\| \mid \text{for all cliff points } \mathbf{f} \in [\mathbf{u}, \mathbf{v}]\right\}.$$

Let $\mathbf{p}_1 \in [\mathbf{n}, \mathbf{m}]$ be chosen so that $\mathbf{p}_1 = \mathbf{n} + 2(\mathbf{k} - \mathbf{u})$ or equivalently, $\mathbf{p}_1 = \mathbf{m} + 2(\mathbf{k} - \mathbf{v})$. It follows from Lemma 4.2.1.2 that the directions $(\mathbf{p}_1 - \mathbf{k}) - (\mathbf{k}^{+} - \mathbf{k})$ and $(\mathbf{p}_1 - \mathbf{k}) - (\mathbf{k}^{-} - \mathbf{k})$ illuminate $W_{[\mathbf{u},\mathbf{v}]}$.

A similar argument to the one used in the proof of Proposition 4.2.2.2.2 can be used to show that there exists a real number $\chi' > 0$ such that the directions $(\mathbf{p}_1 - \mathbf{k}) - (\mathbf{k}^{+} - \mathbf{k})$ and $(\mathbf{p}_1 - \mathbf{k}) - (\mathbf{k}^{-} - \mathbf{k})$ illuminate the set $W_{\ell' \cap \mathrm{relbd}(B)} + \mathrm{B}(\mathbf{o}, \chi)$, which is an open neighbourhood of $W_{[\mathbf{u},\mathbf{v}]}$ on the bd(K).

Then, by Lemma 4.2.2.1.3, Proposition 4.2.2.1.8 and Proposition 4.2.2.1.9, there exists points $\mathbf{a}, \mathbf{b} \in \mathrm{relbd}(B) \cap \left(W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o}, 1)\right)$ such that the points $\mathbf{u}, \mathbf{q}, \mathbf{v}, \mathbf{b}$ and $\mathbf{a}$ follow each other in this order when travelling counter-clockwise on relbd(B) starting at $\mathbf{u}$. Moreover, the line passing through $\mathbf{a}$ and $\mathbf{b}$ is parallel to the support line ℓ'. By Lemma 4.2.2.1.12, the directions $\frac{1}{2}(\mathbf{q} + \mathbf{a}) - \mathbf{b}$, $\frac{1}{2}(\mathbf{q} + \mathbf{b}) - \mathbf{a}$, $\left(1 - \frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}(\mathbf{q} + \mathbf{a}) - \mathbf{b}\right) \pm \mathcal{T}\mathbf{e}_3$ and

$\left(1 - \frac{2(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{b}) - \mathbf{a}\right) \pm \mathcal{T}\mathbf{e}_3$ will illuminate $\mathrm{bd}(K)\backslash\left(W_{\ell' \cap \mathrm{relbd}(B)} + \chi' \mathrm{B}(\mathbf{o},1)\right)$.

This means that the eight directions $\frac{1}{2}(\mathbf{q}+\mathbf{a}) - \mathbf{b}$, $\frac{1}{2}(\mathbf{q}+\mathbf{b}) - \mathbf{a}$, $(\mathbf{p}_1 - \mathbf{k}) - (\mathbf{k}^+ - \mathbf{k})$, $(\mathbf{p}_1 - \mathbf{k}) - (\mathbf{k}^- - \mathbf{k})$, $\left(1 - \frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{a}) - \mathbf{b}\right) \pm \mathcal{T}\mathbf{e}_3$ and $\left(1 - \frac{2(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{b}) - \mathbf{a}\right) \pm \mathcal{T}\mathbf{e}_3$, and will illuminate $\mathrm{bd}(K)$.

4.2.5 Fourth Major Case of Theorem 4.1

In this fourth case, suppose that

(i) $\mathrm{relbd}(B)$ contains at least one side and for each side of $\mathrm{relbd}(B)$, there exists another side parallel to it;

(ii) each side of $\mathrm{relbd}(B)$ is non-degenerate;

(iii) each side of $\mathrm{relbd}(B)$ has length at least $\frac{1}{2}$.

By assumption, $\mathrm{relbd}(B)$ contains at least two sides. Let $[\mathbf{u}, \mathbf{v}]$ be an arbitrary side of $\mathrm{relbd}(B)$ and let $[\mathbf{w}, \mathbf{z}]$ be the side nearest to $[\mathbf{u}, \mathbf{v}]$ when travelling counter-clockwise on $\mathrm{relbd}(B)$. Moreover, let the points $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and $\mathbf{z}$ follow each other in this order when travelling counter-clockwise on $\mathrm{relbd}(B)$, starting at the point $\mathbf{u}$.

4.2.5.1 Suppose that $\mathbf{v} \neq \mathbf{w}$.

Since $[\mathbf{w}, \mathbf{z}]$ is the side nearest to $[\mathbf{u}, \mathbf{v}]$ when travelling counter-clockwise on $\mathrm{relbd}(B)$, $[\mathbf{v}, \mathbf{w}]_B$ does not contain any sides. As mentioned in §4.2.2, smooth points are dense in $\mathrm{relbd}(B)$. So, let $\mathbf{p} \in (\mathbf{v}, \mathbf{w})_B$ be some arbitrary smooth point. Denote the unique supporting line of B at $\mathbf{p}$ by ℓ. The complete antipode $A(\mathbf{p})$ is either a single point or a side of B.

If the complete antipode $A(\mathbf{p})$ is a side of B, then by condition *(i)* there exists another side parallel to it in $\mathrm{relbd}(B)$. It follows from Theorem 2.10.1.2 that the convex body B is supported by exactly two lines, which means that $\mathbf{p}$ must lie on a side in $[\mathbf{v}, \mathbf{w}]_B$. This is a contradiction.

Therefore, $A(\mathbf{p})$ is a single point, which we denote by $\mathbf{q}$. If $\mathbf{q}$ is a ground point, K can be illuminated using the same procedure as in §4.2.2.1. If $\mathbf{q}$ is a cliff point, the same procedure as in §4.2.2.2 can be used to illuminate K.

This takes care of the sub-case where between any two sides in $\text{relbd}(B)$, there is an arc containing no sides.

4.2.5.2 Suppose that $\mathbf{v} = \mathbf{w}$.

It follows that $\text{relbd}(B)$ is composed of only sides.

Lemma 4.2.5.2.1. *B is a polygon.*

Proof. Since $\text{relbd}(B)$ is made up of sides each of length at least $\frac{1}{2}$, $\text{relbd}(B)$ has finitely many sides. These sides are 1-dimensional polytopes and by definition can be expressed as the convex hull of finitely many points. This implies B has finitely many extreme points. Since B is compact and convex, the Krein-Milman Theorem implies that B can be expressed as the convex hull of its extreme points. Therefore, by definition B is a polytope. In particular, B is a polygon since $\dim(B) = \dim\big(\text{aff}(B)\big) = \dim(\mathbb{E}^2 \times \{0\}) = 2$. ■

Moreover, B is a $2n$-gon where $n \geqslant 2$, due to supposition *(i)*. Note that all remaining cases are sub-cases of §4.2.5.2.

4.2.6 First Featured Subcase of Theorem 4.1

Suppose that

(i) $\text{relbd}(B)$ contains at least one side and for each side of $\text{relbd}(B)$, there exists another side parallel to it;

(ii) each side of $\text{relbd}(B)$ is non-degenerate;

(iii) each side of $\text{relbd}(B)$ has length at least $\frac{1}{2}$; and

(iv) B is a quadrilateral.

It follows from (i) & (iv) that B must be a parallelogram. Label the vertices of B by $\mathbf{v}_i$, for $i \in \mathbb{Z}_4$ in a counter-clockwise fashion so that the vertices $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ are adjacent. Since $\text{relbd}(B)$ is a simple closed curve, it should be clear that the sides $[\mathbf{v}_i, \mathbf{v}_{i+1}]$ and $[\mathbf{v}_{i+2}, \mathbf{v}_{i+3}]$ of $\text{relbd}(B)$ are parallel, for any $i \in \mathbb{Z}_4$. In this case, non-adjacent vertices are called *opposite* vertices.

4.2.6.1 Suppose all the vertices of B are cliff points.

Recall that $\mathcal{T} = \max\{\|\mathbf{k}^+ - \mathbf{k}^-\| \mid \mathbf{k} \in K\}$, where $\mathbf{k}^+$ is the endpoint of the non-degenerate line segment $\text{Pr}(\mathbf{k}) \cap K$ lying in H_+ and $\mathbf{k}^-$ is the other endpoint of that line segment lying in H_-. Then, the eight vectors $(\mathbf{v}_i - \mathbf{v}_{i+2}) \pm \mathcal{T}\mathbf{e}_3$, $(\mathbf{v}_{i+1} - \mathbf{v}_{i+3}) \pm \mathcal{T}\mathbf{e}_3$, $(\mathbf{v}_{i+2} - \mathbf{v}_i) \pm \mathcal{T}\mathbf{e}_3$ and $(\mathbf{v}_{i+3} - \mathbf{v}_{i+1}) \pm \mathcal{T}\mathbf{e}_3$ illuminate $\text{bd}(K)$.

4.2.6.2 Suppose that one vertex of B is a ground point and that the other three vertices of B are cliff points.

Let $\mathbf{q}$ denote the vertex of B which is a ground point and denote its opposite vertex by $\mathbf{p}$. Let ℓ' be a supporting line of B at $\mathbf{q}$ such that ℓ' does not support any sides of B. There exists supporting line of B at $\mathbf{p}$ parallel to ℓ'; denote it by ℓ. However, note that ℓ and ℓ' are not unique. A nearly identical proof to the one used in §4.2.2.1 can be used to show that the seven directions $\mathbf{p} - \mathbf{q}$, $\frac{1}{2}(\mathbf{q} + \mathbf{a}) - \mathbf{b}$, $\frac{1}{2}(\mathbf{q} + \mathbf{b}) - \mathbf{a}$, $\left(1 - \frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}(\mathbf{q} + \mathbf{a}) - \mathbf{b}\right) \pm \mathcal{T}\mathbf{e}_3$ and $\left(1 - \frac{2(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}(\mathbf{q} + \mathbf{b}) - \mathbf{a}\right) \pm \mathcal{T}\mathbf{e}_3$ illuminate $\text{bd}(K)$.

4.2.6.3 Suppose two adjacent vertices of B are ground points and that the other two vertices of B are cliff points.

Then, there exists some $i \in \mathbb{Z}_4$ such that the vertices $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ are ground points. The eight vectors $(\mathbf{v}_{i+2} - \mathbf{v}_i)$, $(\mathbf{v}_{i+3} - \mathbf{v}_i)$, $(\mathbf{v}_{i+3} - \mathbf{v}_i) \pm \mathcal{T}\mathbf{e}_3$, $(\mathbf{v}_i - \mathbf{v}_{i+2}) \pm \mathcal{T}\mathbf{e}_3$ and $(\mathbf{v}_{i+1} - \mathbf{v}_{i+3}) \pm \mathcal{T}\mathbf{e}_3$ illuminate $\text{bd}(K)$.

4.2.6.4 Suppose two opposite vertices of B are ground points.

Then, there exists some $i \in \mathbb{Z}_4$ such that the vertices $\mathbf{v}_i$ and $\mathbf{v}_{i+2}$ are ground points. The eight vectors $\mathbf{v}_{i+2}-\mathbf{v}_i$, $\mathbf{v}_i-\mathbf{v}_{i+2}$, $\mathbf{v}_{i+3}-\mathbf{v}_i$, $\mathbf{v}_i-\mathbf{v}_{i+3}$, $(\mathbf{v}_{i+3}-\mathbf{v}_i)\pm\mathcal{T}\mathbf{e}_3$ and $(\mathbf{v}_i-\mathbf{v}_{i+3})\pm\mathcal{T}\mathbf{e}_3$ illuminate $\mathrm{bd}(K)$.

4.2.7 Second Featured Subcase of Theorem 4.1

Suppose that

(i) $\mathrm{relbd}(B)$ contains at least one side and for each side of $\mathrm{relbd}(B)$, there exists another side parallel to it;

(ii) each side of $\mathrm{relbd}(B)$ is non-degenerate;

(iii) each side of $\mathrm{relbd}(B)$ has length at least $\frac{1}{2}$; and

(iv) B is a $2n$-gon, for any $n \geqslant 4$.

4.2.7.1 Suppose that two consecutive vertices of B are cliff points.

Let $\mathbf{u}$ and $\mathbf{v}$ denote two consecutive vertices of B which are cliff points. Denote the supporting line of B at the side $[\mathbf{u},\mathbf{v}]$ by ℓ' and let $[\mathbf{w},\mathbf{z}]$ be the side parallel to $[\mathbf{u},\mathbf{v}]$ where the points $\mathbf{u},\mathbf{v},\mathbf{w}$ and $\mathbf{z}$ follow each other in this order when travelling counter-clockwise on $\mathrm{relbd}(B)$ and starting at the point $\mathbf{u}$. Let $\mathbf{q}=\frac{1}{2}(\mathbf{u}+\mathbf{v})$ and denote the supporting line of B at $[\mathbf{w},\mathbf{z}]$ parallel to ℓ' by ℓ. Moreover, let the supporting lines of the sides adjacent to $[\mathbf{u},\mathbf{v}]$ by $\ell^\dagger$ and $\ell^\ddagger$. The lines $\ell^\dagger$ and $\ell^\ddagger$ are not parallel; denote their intersection point by $\mathbf{m}$ and notice that $\mathbf{m}\notin B$. Then, there exists $\chi'>0$ such that the directions $(\mathbf{q}-\mathbf{m})\pm\mathcal{T}\mathbf{e}_3$ illuminate the open neighbourhood $W_{\ell'\cap\mathrm{relbd}(B)}+\chi'\mathrm{B}(\mathbf{o},1)$ on the $\mathrm{bd}(K)$.

By Lemma 4.2.2.1.3, Proposition 4.2.2.1.8 and Proposition 4.2.2.1.9, there exists points $\mathbf{a},\mathbf{b}\in\mathrm{relbd}(B)\cap\left(W_{\ell'\cap\mathrm{relbd}(B)}+\chi'\mathrm{B}(\mathbf{o},1)\right)$ such that the points $\mathbf{u},\mathbf{q},\mathbf{v},\mathbf{b},\mathbf{w},\mathbf{z}$ and $\mathbf{a}$ follow each other in this order when travelling counter-clockwise on $\mathrm{relbd}(B)$ starting at $\mathbf{u}$. Moreover, the line passing through $\mathbf{a}$ and $\mathbf{b}$ is parallel to the support line ℓ'. By Lemma 4.2.2.1.12, the directions $\frac{1}{2}(\mathbf{q}+\mathbf{a})-\mathbf{b}$, $\frac{1}{2}(\mathbf{q}+\mathbf{b})-\mathbf{a}$, $\left(1-\frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{a})-\mathbf{b}\right)\pm\mathcal{T}\mathbf{e}_3$ and $\left(1-\frac{2(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{b})-\mathbf{a}\right)\pm\mathcal{T}\mathbf{e}_3$ will illuminate $\mathrm{bd}(K)\backslash\left(W_{\ell'\cap\mathrm{relbd}(B)}+\chi'\mathrm{B}(\mathbf{o},1)\right)$.

This means that the eight directions $(\mathbf{q}-\mathbf{m})\pm\mathcal{T}\mathbf{e}_3$, $\frac{1}{2}(\mathbf{q}+\mathbf{a})-\mathbf{b}$, $\frac{1}{2}(\mathbf{q}+\mathbf{b})-\mathbf{a}$, $\left(1-\frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{a})-\mathbf{b}\right)\pm\mathcal{T}\mathbf{e}_3$, $\left(1-\frac{2(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{b})-\mathbf{a}\right)\pm\mathcal{T}\mathbf{e}_3$ illuminate $\mathrm{bd}(K)$.

4.2.7.2 Suppose there exists a pair of parallel sides $[\mathbf{u},\mathbf{v}]$ and $[\mathbf{w},\mathbf{z}]$ of $\mathrm{relbd}(B)$, where the vertices u, v, w and z follow each other in this order when starting at the vertex u and moving counter-clockwise on the $\mathrm{relbd}(B)$, such that either u and w are ground points or v and z are ground points.

Suppose without loss of generality that the vertices $\mathbf{u}$ and $\mathbf{w}$ are ground points. Let ℓ and ℓ' be the supporting lines of B chosen so that $\ell\cap B=\{\mathbf{u}\}$ and $\ell'\cap B=\{\mathbf{w}\}$. Moreover, suppose that $\ell=\left\{\mathbf{u}+\lambda\tilde{\mathbf{d}}\mid\lambda\in\mathbb{R}\right\}$. Then, the eight directions $\pm\tilde{\mathbf{d}}$, $\tilde{\mathbf{d}}\pm\mathcal{T}\mathbf{e}_3$, $-\tilde{\mathbf{d}}\pm\mathcal{T}\mathbf{e}_3$, $\mathbf{w}-\mathbf{u}$ and $\mathbf{u}-\mathbf{w}$ illuminate $\mathrm{bd}(K)$.

4.2.7.3 Suppose that vertices of $\mathrm{relbd}(B)$ alternate between cliff and ground points.

Note that this case is distinct from §4.2.7.2, only if n in (iv) is odd. Namely, §4.2.7.2 does not include this case if B is a decagon, or if B a 14-gon, etcetera.

4.2.8 Third Featured Subcase of Theorem 4.1

Suppose that

(i) $\mathrm{relbd}(B)$ contains at least one side and for each side of $\mathrm{relbd}(B)$, there exists another side parallel to it;

(ii) each side of $\mathrm{relbd}(B)$ is non-degenerate;

(iii) each side of $\mathrm{relbd}(B)$ has length at least $\frac{1}{2}$; and

(iv) B is a hexagon.

4.2.8.1 Suppose that two consecutive vertices of B are cliff points.

In this case, K can be illuminated using the exact same procedure as in §4.2.7.1

4.2.8.2 Suppose there exists a pair of parallel sides $[\mathbf{u}, \mathbf{v}]$ and $[\mathbf{w}, \mathbf{z}]$ of $\operatorname{relbd}(B)$, where the vertices u, v, w and z follow each other in this order when starting at the vertex u and moving counter-clockwise on the $\operatorname{relbd}(B)$, such that either u and w are ground points or v and z are ground points.

In this case, K can be illuminated using the exact same procedure as in §4.2.7.2.

4.2.8.3 Suppose that vertices of $\operatorname{relbd}(B)$ alternate between cliff and ground points.

Let H_0 be a regular hexagon. Label its vertices by $\mathbf{v}_i$ for $i \in \mathbb{Z}_6$ in a counter- clockwise fashion such that $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ are consecutive vertices. Choose one pair of parallel edges from H_0, say $[\mathbf{v}_0, \mathbf{v}_1]$ and $[\mathbf{v}_3, \mathbf{v}_4]$. Let H be the hexagon obtained from taking the convex hull of the point set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_0 + \lambda(\mathbf{v}_0 - \mathbf{v}_1), \mathbf{v}_4 + \lambda(\mathbf{v}_0 - \mathbf{v}_1), \mathbf{v}_5 + \lambda(\mathbf{v}_0 - \mathbf{v}_1)\}$, for some scalar $\lambda \geqslant 0$. Notice that the length of exactly one pair of parallel sides from the hexagon H_0 are scaled by $\lambda \geqslant 0$ and the rest of the sides have the same length.

Case 1: Suppose that B is not an affine image of H. The following observation plays an important role in the proof of Lemma 4.2.8.3.2.

Proposition 4.2.8.3.1. *Let H be a convex hexagon in $\mathbb{E}^2 \times \{\mathbf{o}\}$ with the property that for each of its sides, it has another side parallel to it. Then, there exists two triangles T_1 and T_2 such that $H = T_1 \cap T_2$ where $T_2 = \lambda T_1 + \mathbf{t}$ for some $\lambda < 0$ and vector $\mathbf{t} \in \mathbb{E}^3$.*

Proof. Let the sides of H be labelled by S_i, where $i \in \mathbb{Z}_6$, so that the sides S_i and S_{i+1} are adjacent. Let the vertices of H be labelled by $\mathbf{v}_i$, where $i \in \mathbb{Z}_6$, such that $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ are consecutive vertices and $S_i = [\mathbf{v}_i, \mathbf{v}_{i+1}]$.

Claim: The sides S_i and S_{i+3} are parallel: adjacent sides are not parallel and there does not exist any $i \in Z_6$ such that the sides S_i and S_{i+2} are parallel.

Suppose for a contradiction that there exists a pair of adjacent sides S_i and S_{i+1} that are parallel. This means that there exists $\xi \in \mathbb{R}$ such that $\mathbf{v}_{i+1} - \mathbf{v}_i = \xi (\mathbf{v}_{i+2} - \mathbf{v}_{i+1})$. Both sides share the vertex $\mathbf{v}_{i+1}$. Let ℓ_i and ℓ_{i+1} denote the supporting lines of H at S_i and S_{i+1}, respectively. Notice that

$$\begin{aligned} \ell_i &= \{\mathbf{v}_{i+1} + \lambda (\mathbf{v}_{i+1} - \mathbf{v}_i) \mid \lambda \in \mathbb{R}\} \\ &= \{\mathbf{v}_{i+1} + \lambda\xi (\mathbf{v}_{i+2} - \mathbf{v}_{i+1}) \mid \lambda\xi \in \mathbb{R}\} = \ell_{i+1}. \end{aligned}$$

Since the supporting lines ℓ_i and ℓ_{i+1} are not distinct, this means that either $S_i = S_{i+1}$ or $S_i \cup S_{i+1}$ is a side of H. In either case, this would imply that H has only five sides, which is a contradiction.

Suppose for a contradiction that there exists $i \in \mathbb{Z}_6$ such that S_i and S_{i+2} are parallel. Let ℓ_i and ℓ_{i+2} denote the supporting lines of H through the sides S_i and S_{i+2}, respectively. Then, ℓ_i and ℓ_{i+2} are parallel. It follows from Theorem 2.10.1.2 that no other supporting line of H is parallel to ℓ_i and ℓ_{i+2}. Therefore, one of the sides S_{i+3}, S_{i+4} or S_{i+5} is parallel to S_{i+1}, since H has the property that for each of its sides, there exists another side of H parallel to it. Moreover, the remaining two sides of H must be parallel to each other. Therefore, if S_{i+3} were parallel to S_{i+1}, the sides S_{i+4} and S_{i+5} would have to be parallel to each other. However, it was shown above that adjacent sides cannot be parallel. So, this case cannot occur. Likewise, if S_{i+5} were parallel to S_{i+1}, then the adjacent sides S_{i+3} and S_{i+4} would have to be parallel to each other, which is not possible.

If the side S_{i+4} is parallel to S_{i+1}, then the sides S_{i+3} and S_{i+5} must be parallel to each other. This means that there exists some $\xi' \in \mathbb{R}$ such that and $\mathbf{v}_{i+3} - \mathbf{v}_{i+4} = \xi' (\mathbf{v}_{i+5} - \mathbf{v}_i)$. It follows from Theorem 2.10.1.3 that H must lie between its supporting lines ℓ_i and ℓ_{i+2}. The vertices $\mathbf{v}_{i+1}$ and $\mathbf{v}_{i+2}$ of the side S_{i+1} lie on ℓ_i and ℓ_{i+2}, respectively. Also, the vertices $\mathbf{v}_i$ and $\mathbf{v}_{i+3}$ lie on the lines ℓ_i and ℓ_{i+2}; therefore, the vertices $\mathbf{v}_{i+4}$ and $\mathbf{v}_{i+5}$ must lie strictly between the supporting lines ℓ_i and ℓ_{i+2}, otherwise H would be a quadrilateral. This means that there exists $0 < \tilde{\xi} < 1$ such that $\mathbf{v}_{i+5} - \mathbf{v}_{i+4} = \tilde{\xi}\,(\mathbf{v}_{i+1} - \mathbf{v}_{i+2})$. Therefore, the supporting line of B at S_{i+3} is of the form

$$\begin{aligned}\ell_{i+3} &= \{\mathbf{v}_{i+4} + \lambda'\xi'\,(\mathbf{v}_i - \mathbf{v}_{i+5}) \mid \lambda'\xi' \in \mathbb{R}\} \\ &= \left\{\mathbf{v}_i + \tilde{\xi}\,(\mathbf{v}_{i+2} - \mathbf{v}_{i+1}) + (1 - \lambda'\xi')\,(\mathbf{v}_{i+5} - \mathbf{v}_i) \mid 1 - \lambda'\xi' \in \mathbb{R}\right\}.\end{aligned}$$

Notice that the point on the supporting line ℓ_{i+3},

$$\begin{aligned}\mathbf{v}_{i+4} + \frac{1}{\xi'}\xi'\,(\mathbf{v}_i - \mathbf{v}_{i+5}) &= \mathbf{v}_i + \tilde{\xi}\,(\mathbf{v}_{i+1} - \mathbf{v}_{i+2}) + (\mathbf{v}_{i+5} - \mathbf{v}_i) + (\mathbf{v}_i - \mathbf{v}_{i+5}) \\ &= \mathbf{v}_i + \tilde{\xi}\,(\mathbf{v}_{i+1} - \mathbf{v}_{i+2})\,,\end{aligned}$$

also belongs to $(\mathbf{v}_i, \mathbf{v}_{i+2})$. By the convexity of H and Proposition 2.10.6,

$$\operatorname{conv}\{\mathbf{v}_i, \mathbf{v}_{i+1}, \mathbf{v}_{i+2}, \mathbf{v}_{i+3}\} \subseteq H$$

and hence, $[\mathbf{v}_i, \mathbf{v}_{i+2}] \subseteq H$. The line ℓ_{i+3} determines two open half-spaces:

$$\ell_{i+3}^{+} = \left\{\mathbf{z} \in \mathbb{E}^3 \mid \mathbf{z} = \mathbf{v}_i + \widehat{\lambda}\,(\mathbf{v}_{i+2} - \mathbf{v}_{i+1}) + (1 - \lambda\xi')\,(\mathbf{v}_{i+5} - \mathbf{v}_i)\,,\ \widehat{\lambda} > \tilde{\xi},\ 1 - \lambda'\xi' \in \mathbb{R}\right\}$$

and

$$\ell_{i+3}^{-} = \left\{\mathbf{z} \in \mathbb{E}^3 \mid \mathbf{z} = \mathbf{v}_i + \widehat{\lambda}\,(\mathbf{v}_{i+2} - \mathbf{v}_{i+1}) + (1 - \lambda\xi')\,(\mathbf{v}_{i+5} - \mathbf{v}_i)\,,\ \widehat{\lambda} < \tilde{\xi},\ 1 - \lambda'\xi' \in \mathbb{R}\right\}.$$

Since $0 < \tilde{\xi} < 1$, the point $\mathbf{v}_{i+2}$ of H lies in the open half-space ℓ_{i+3}^{+} and the point $\mathbf{v}_i$ of H lies in the open half-space ℓ_{i+3}^{-}. This means that ℓ_{i+3} strictly separates two points of H, which contradicts that it is a supporting hyperplane of H. Therefore, the sides S_i and S_{i+2} cannot be parallel.

Hence, the sides S_i and S_{i+3} are parallel for any $i \in \mathbb{Z}_6$.

■

Lemma 4.2.8.3.2. *Let B be a hexagon such that*

(i) for any side of B, there exists another side of B parallel to it;

(ii) *B is not the affine image of a hexagon obtained by scaling the lengths of exactly one pair of parallel sides from a regular hexagon by a scalar $\lambda \geqslant 0$ while preserving the other edge lengths.*

Then, $\mathrm{relint}(B)$ *contains a line segment* $[\mathbf{n}, \mathbf{m}]$ *such that* $\mathbf{m} - \mathbf{n} = 2(\mathbf{v} - \mathbf{u})$*, for some side* $[\mathbf{u}, \mathbf{v}] \subseteq \mathrm{relbd}(B)$*.*

Proof. It follows from Proposition 4.2.8.3.1 that there exist two triangles T_1 and T_2 such that $B = T_1 \cap T_2$ and $T_2 = \lambda T_1 + \mathbf{t}$ for some $\lambda < 0$ and $\mathbf{t} \in \mathbb{E}^n$. If the triangles T_1 and T_2 are not regular, apply an affine transformation to K so that the triangles T_1 and T_2 are regular. Denote the center of T_1 by $\mathbf{z}$ and its vertices by $\mathbf{a}_1, \mathbf{b}_1$ and $\mathbf{c}_1$ so that they follow each other in this order when travelling counter-clockwise on the relative boundary of T_1. Without loss of generality suppose that $T_2 = \lambda T_1 + \mathbf{t}$, for some $-1 \leqslant \lambda \leqslant 0$ and label its vertices by $\mathbf{a}_2, \mathbf{b}_2$ and $\mathbf{c}_2$ so that $\mathbf{a}_2 = \lambda \mathbf{a}_1 + \mathbf{t}$, $\mathbf{b}_2 = \lambda \mathbf{b}_1 + \mathbf{t}$ and $\mathbf{c}_2 = \lambda \mathbf{c}_1 + \mathbf{t}$. The triangle $T_0 = -T + 2\mathbf{z}$ has the same center as T_1; denote its vertices by $\mathbf{a}_0, \mathbf{b}_0$ and $\mathbf{c}_0$ so that $\mathbf{a}_0 = -\mathbf{a}_1 + 2\mathbf{z}$, $\mathbf{b}_0 = -\mathbf{b}_1 + 2\mathbf{z}$ and $\mathbf{c}_0 = -\mathbf{c}_1 + 2\mathbf{z}$. Note that T_2 cannot have the same center as T_1 and T_0, otherwise the hexagon $B = T_1 \cap T_2$ would be regular and this would violate condition *(ii)* of Lemma 4.2.8.3.2. In particular, this means that $T_2 \nsubseteq T_0$. One of the sides of T_2 intersects the regular hexagon $T_1 \cap T_0$. Suppose without loss of generality that the side $[\mathbf{a}_2, \mathbf{b}_2] \cap (T_1 \cap T_0) \neq \varnothing$. The line $\{\mathbf{a}_2 + \mu(\mathbf{b}_2 - \mathbf{a}_2) \mid \mu \in \mathbb{R}\}$ intersects the line segments $[\mathbf{a}_0, \mathbf{b}_1]$ and $[\mathbf{b}_0, \mathbf{a}_1]$ of the parallelogram $\mathrm{conv}\{\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_0, \mathbf{b}_0\}$; denote the points of intersection by $\mathbf{a}'$ and $\mathbf{b}'$, respectively. Suppose without loss of generality that $\|\mathbf{a}_2 - \mathbf{z}\| \leqslant \|\mathbf{b}_2 - \mathbf{z}\|$.

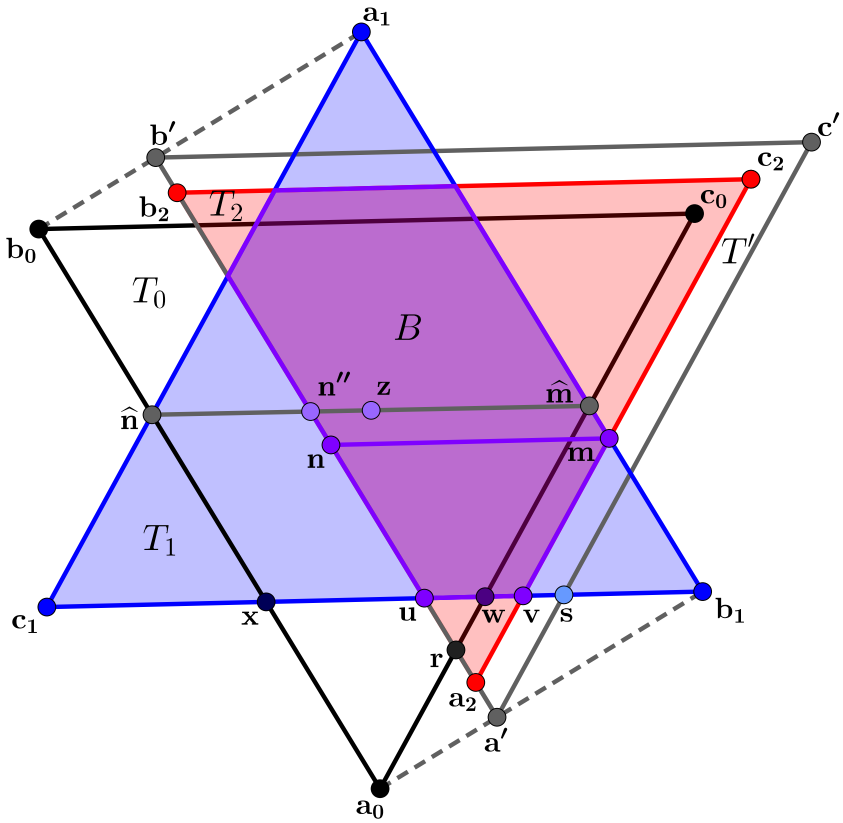


Figure 4.12

Observe that $\mathbf{a}_2 \in \operatorname{conv}\{\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_0, \mathbf{b}_0\}$. Let $\{\mathbf{x}\} = [\mathbf{b}_0, \mathbf{a}_0] \cap [\mathbf{c}_1, \mathbf{b}_2]$. Then, $\mathbf{a}$ lies in the triangle $\operatorname{conv}\{\mathbf{x}, \mathbf{a}_0, \mathbf{b}_1\}$. Let the triangle T' with vertices $\mathbf{a}', \mathbf{b}'$ and $\mathbf{c}'$ denote the translate of T_0 by $\mathbf{a}' - \mathbf{a}_0$. Label the intersections of various line segments in the following way:

$$\{\mathbf{u}\} = [\mathbf{b}_1, \mathbf{c}_1] \cap [\mathbf{a}', \mathbf{b}']\,; \quad \{\mathbf{w}\} = [\mathbf{b}_1, \mathbf{c}_1] \cap [\mathbf{a}_0, \mathbf{c}_0]\,; \quad \{\mathbf{v}\} = [\mathbf{b}_1, \mathbf{c}_1] \cap [\mathbf{a}_2, \mathbf{c}_2]\,;$$

$$\{\mathbf{s}\} = [\mathbf{b}_1, \mathbf{c}_1] \cap [\mathbf{a}', \mathbf{c}']\,; \quad \{\mathbf{r}\} = [\mathbf{a}_0, \mathbf{c}_0] \cap [\mathbf{a}', \mathbf{b}']\,; \quad \{\widehat{\mathbf{m}}\} = [\mathbf{a}_1, \mathbf{b}_1] \cap [\mathbf{a}_0, \mathbf{c}_0]\,;$$

$$\{\mathbf{m}\} = [\mathbf{a}_1, \mathbf{b}_1] \cap [\mathbf{a}_2, \mathbf{c}_2]\,; \quad \{\widehat{\mathbf{n}}\} = [\mathbf{a}_1, \mathbf{c}_1] \cap [\mathbf{a}_0, \mathbf{b}_0]\,; \quad \{\mathbf{n}''\} = [\mathbf{p}_2, \mathbf{q}_2] \cap [\mathbf{a}', \mathbf{b}']\,.$$

Let $\mathbf{n} \in [\mathbf{a}_2, \mathbf{b}_2]$ be chosen so that the chord $[\mathbf{n}, \mathbf{m}]$ of $B = T_1 \cap T_2$ is parallel to its side $[\mathbf{u}, \mathbf{v}]$. The chord $[\mathbf{n}, \mathbf{m}]$ is the longest chord of B. Denote $\|\mathbf{x} - \mathbf{u}\|$ by $2\varphi < \|\mathbf{x} - \mathbf{b}_1\|$. Notice that $\|\mathbf{w} - \mathbf{s}\| = \varphi$. Therefore, $2 = \dfrac{\|\widehat{\mathbf{m}} - \widehat{\mathbf{n}}\|}{\|\mathbf{w} - \mathbf{x}\|} = \dfrac{\|\widehat{\mathbf{m}} - \mathbf{n}''\| + \|\mathbf{n}'' - \widehat{\mathbf{n}}\|}{\|\mathbf{w} - \mathbf{x}\|} = \dfrac{\|\mathbf{m} - \mathbf{n}\| + 2\varphi}{\varphi + \|\mathbf{s} - \mathbf{u}\}}$. This implies that $\mathbf{m}-\mathbf{n} = 2(\mathbf{s}-\mathbf{u})$. Also, $\|\mathbf{s}-\mathbf{u}\| > \|\mathbf{v}-\mathbf{u}\|$. This means that $\mathbf{m}-\mathbf{n} > 2(\mathbf{v}-\mathbf{u})$. Finally, $(\mathbf{n},\mathbf{m}) \subseteq \operatorname{int}(K)$.

■

Recall that $[\mathbf{u},\mathbf{v}]$ is non-degenerate and in fact, either $\mathbf{u}$ or $\mathbf{v}$ is a cliff point. Let $\mathbf{k} \in [\mathbf{u},\mathbf{v}]$ be chosen so that

$$\left\|\mathbf{k}^+ - \mathbf{k}^-\right\| = \max\left\{\left\|\mathbf{f}^+ - \mathbf{f}^-\right\| \mid \text{for all cliff points } \mathbf{f} \in [\mathbf{u},\mathbf{v}]\right\}.$$

Let $\mathbf{p_1} \in [\mathbf{m},\mathbf{n}]$ be chosen so that $\mathbf{p_1} - \mathbf{n} = 2(\mathbf{k}-\mathbf{u})$ and $\mathbf{m} - \mathbf{p_1} = 2(\mathbf{v}-\mathbf{k})$. Then, by Lemma 4.2.1.2, the directions $(\mathbf{p_1}-\mathbf{k})-(\mathbf{k}^+-\mathbf{k})$ and $(\mathbf{p_1}-\mathbf{k})-(\mathbf{k}^--\mathbf{k})$ illuminate $W_{[\mathbf{u},\mathbf{v}]}$.

By Proposition 4.2.2.2.2, there exists a real number $\chi' > 0$ such that the directions $(\mathbf{p_1}-\mathbf{k})-(\mathbf{k}^+-\mathbf{k})$ and $(\mathbf{p_1}-\mathbf{k})-(\mathbf{k}^--\mathbf{k})$ illuminate an open neighbourhood of $W_{[\mathbf{u},\mathbf{v}]}$ on the boundary of K, $W_{[\mathbf{u},\mathbf{v}]} + \chi'\mathrm{B}(\mathbf{o},1)$.

It follows from Lemma 4.2.2.1.3, Proposition 4.2.2.1.8 and Proposition 4.2.2.1.9 that there exists points $\mathbf{a}$ and $\mathbf{b}$ in this open neighbourhood such that the line between them is parallel to the supporting line at the side $[\mathbf{u},\mathbf{v}]$. Let $\mathbf{q} = \frac{1}{2}(\mathbf{u}+\mathbf{v})$. Then, by Lemma 4.2.2.1.12, the six directions $\frac{1}{2}(\mathbf{q}+\mathbf{a})-\mathbf{b}$, $\frac{1}{2}(\mathbf{q}+\mathbf{b})-\mathbf{a}$, $\left(1-\frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{a})-\mathbf{b}\right) \pm \mathcal{T}\mathbf{e}_3$, and $\left(1-\frac{2(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}(\mathbf{q}+\mathbf{b})-\mathbf{a}\right) \pm \mathcal{T}\mathbf{e}_3$ will illuminate $\operatorname{bd}(K)\backslash\left(W_{[\mathbf{u},\mathbf{v}]} + \chi'\mathrm{B}(\mathbf{o},1)\right)$.

Case 2: Suppose that B is an affine image of H.

Then, there exists an affine transformation $T : \mathbb{E}^n \to \mathbb{E}^n$ such that $B = T(H)$. It follows from *(vi)*,*(v)* and *(vi)* of Properties 2.4.8 that B is a hexagon with the property that both sides from a pair of parallel sides have the same length. Label the vertices of B by $\mathbf{b}_i$ for $i \in \mathbb{Z}_6$ in a counter-clockwise fashion such that $\mathbf{b}_i$ and $\mathbf{b}_{i+1}$ are consecutive vertices. Choose one pair of parallel sides of B satisfying $\min\{\|\mathbf{b}_i - \mathbf{b}_{i+1}\| \mid i \in \mathbb{Z}_6\}$ and denote them by $[\mathbf{u},\mathbf{v}]$ and $[\mathbf{n},\mathbf{m}]$ where $\mathbf{v},\mathbf{u},\mathbf{n}$ and $\mathbf{m}$ follow each other in this order when starting at $\mathbf{v}$ and travelling counter-clockwise on $\operatorname{relbd}(B)$. Since the vertices of $\operatorname{relbd}(B)$ alternate between ground and cliff points, one may suppose without loss of generality that $\mathbf{u}$ and $\mathbf{n}$ are cliff points while $\mathbf{v}$ and $\mathbf{m}$ are ground points. Let $\mathbf{k} \in [\mathbf{n}, \mathbf{m})$ be the cliff point chosen so that

$$\left\|\mathbf{k}^{+} - \mathbf{k}^{-}\right\| = \max\left\{\left\|\mathbf{f}^{+} - \mathbf{f}^{-}\right\| \mid \text{for all cliff points } \mathbf{f} \in [\mathbf{n}, \mathbf{m})\right\}.$$

It follows that there exists some $0 \leqslant \vartheta < 1$ such that $\mathbf{k} = \vartheta\mathbf{m} + (1 - \vartheta)\,\mathbf{n}$. Let $\mathbf{z} \in [\mathbf{u}, \mathbf{v})$ be chosen so that $\mathbf{z} = \vartheta\mathbf{v} + (1 - \vartheta)\,\mathbf{u}$.

Denote the supporting line of B at the side $[\mathbf{u}, \mathbf{v}]$ by $\ell^{\dagger}$ and denote the supporting line of B at the side $[\mathbf{n}, \mathbf{m}]$, which parallel to it, by $\ell^{\ddagger}$. It follows from Theorem 2.10.1.3 that B lies between $\ell^{\dagger}$ and $\ell^{\ddagger}$. In particular, the other two vertices of B must lie strictly between $\ell^{\dagger}$ and $\ell^{\ddagger}$, otherwise B would be a quadrilateral.

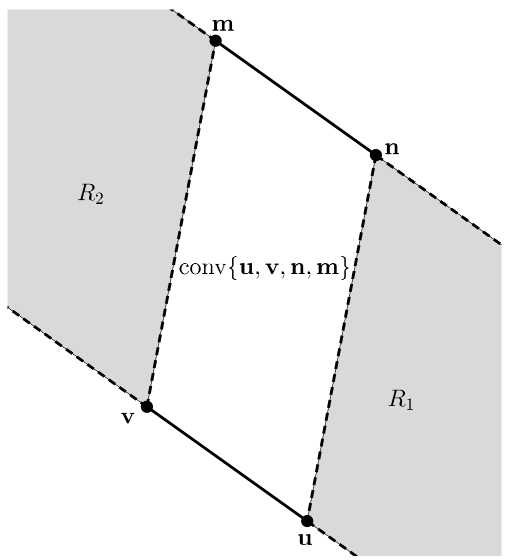


Figure 4.13: One of the remaining two vertices of B must lie in the region R_1 and the other must lie in the region R_2.

By Proposition 2.10.6, $\operatorname{conv}\{\mathbf{u}, \mathbf{v}, \mathbf{n}, \mathbf{m}\} \subseteq B$. The other two vertices of B cannot lie in or on $\operatorname{conv}\{\mathbf{u}, \mathbf{v}, \mathbf{n}, \mathbf{m}\}$, otherwise B could not be a convex hexagon. Moreover, it follows from the Claim in Proposition 4.2.8.3.1 that neither the closed segment $[\mathbf{m}, \mathbf{v}]$ nor the closed segment $[\mathbf{u}, \mathbf{n}]$ can be sides of B; namely, one of the remaining two vertices must lie to the right of the line segment $[\mathbf{u}, \mathbf{n}]$ when travelling counter-clockwise on $\operatorname{relbd}(B)$ and the other vertex must lie to the right of the line segment $[\mathbf{m}, \mathbf{v}]$ when travelling counter-clockwise on relbd(B). This together with Lemma 2.10.13 implies that $(\mathbf{m}, \mathbf{v}) \subseteq \text{relint}(B)$ and $(\mathbf{u}, \mathbf{n}) \subseteq \text{relint}(B)$.

Denote the vertex of B between $\mathbf{u}$ and $\mathbf{n}$ by $\mathbf{d}$ and denote the vertex B between $\mathbf{m}$ and $\mathbf{v}$ by $\mathbf{c}$. Then, there exists $\kappa > 0$ and $0 < \psi < 1$ such that

$$\mathbf{c} = \mathbf{v} + \kappa (\mathbf{v} - \mathbf{u}) + \psi (\mathbf{m} - \mathbf{v}).$$

Since all parallel sides have the same length, it follows that $\mathbf{m} - \mathbf{n} = \mathbf{v} - \mathbf{u}$ and $\mathbf{c} - \mathbf{v} = \mathbf{n} - \mathbf{d}$. Therefore, $\mathbf{m} - \mathbf{v} = \mathbf{n} - \mathbf{u}$. Also,

$$\begin{aligned} \mathbf{d} &= \mathbf{n} + (\mathbf{v} - \mathbf{c}) \\ &= \mathbf{u} + (\mathbf{n} - \mathbf{u}) + \kappa (\mathbf{u} - \mathbf{v}) + \psi (\mathbf{v} - \mathbf{m}) \\ &= \mathbf{u} + (1 - \psi)(\mathbf{n} - \mathbf{u}) + \kappa (\mathbf{u} - \mathbf{v}). \end{aligned}$$

The point $\mathbf{k} + \frac{(1-\vartheta)(1-\psi)}{\kappa} (\mathbf{k} - \mathbf{z})$ is where the line passing through the points $\mathbf{k}$ and $\mathbf{z}$, $\{\mathbf{z} + \lambda (\mathbf{k} - \mathbf{z}) \mid \lambda \in \mathbb{R}\}$, intersects the line passing through the points $\mathbf{m}$ and $\mathbf{c}$. To see this, first notice that

$$\begin{aligned} \mathbf{k} - \mathbf{z} &= \mathbf{n} + \vartheta (\mathbf{m} - \mathbf{n}) - \mathbf{n} + \vartheta (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{n} - \mathbf{u} + \vartheta (\mathbf{v} - \mathbf{u}) + \vartheta (\mathbf{u} - \mathbf{v}) = \mathbf{n} - \mathbf{u} = \mathbf{m} - \mathbf{v}. \end{aligned}$$

Then, observe that $\mathbf{c} = \mathbf{m} + \kappa (\mathbf{v} - \mathbf{u}) + (1 - \psi)(\mathbf{m} - \mathbf{v})$. Re-arrange the equation to get that

$$\begin{aligned} \mathbf{m} - \mathbf{v} &= \frac{1}{1-\psi} (\mathbf{m} - \mathbf{c}) + \frac{\kappa}{1-\psi} (\mathbf{v} - \mathbf{u}) \\ &= \frac{1}{1-\psi} (\mathbf{m} - \mathbf{c}) + \frac{\kappa}{1-\psi} (\mathbf{m} - \mathbf{n}). \end{aligned} \tag{4.33}$$

Finally, observe that

$$\begin{aligned} &\mathbf{k} + \frac{(1-\vartheta)(1-\psi)}{\kappa} (\mathbf{k} - \mathbf{z}) \qquad (4.34) \\ &= \mathbf{m} + (1-\vartheta)(\mathbf{n} - \mathbf{m}) + \frac{(1-\vartheta)(1-\psi)}{\kappa} \left(\frac{1}{1-\psi} (\mathbf{m} - \mathbf{c}) + \frac{\kappa}{1-\psi} (\mathbf{m} - \mathbf{n}) \right) \end{aligned}$$

$$= \mathbf{m} + (1-\vartheta)(\mathbf{n}-\mathbf{m}) + (1-\vartheta)(\mathbf{m}-\mathbf{n}) + \frac{1-\vartheta}{\kappa}(\mathbf{m}-\mathbf{c})$$
$$= \mathbf{m} + \frac{1-\vartheta}{\kappa}(\mathbf{m}-\mathbf{c}) \in \{\mathbf{c} + \lambda'(\mathbf{m}-\mathbf{c}) \mid \lambda' \in \mathbb{R}\}. \tag{4.35}$$

Claim 2: For any $\nu > \dfrac{(1-\vartheta)(1-\psi)}{\kappa}$, the ray through the point $\mathbf{k} + \nu(\mathbf{k}-\mathbf{z})$ with direction $\mathbf{m} + \nu'(\mathbf{c}-\mathbf{m}) - \big(\mathbf{k} + \nu(\mathbf{k}-\mathbf{z})\big)$ intersects relint(B) for $0 \leqslant \nu' < 1$ and only intersects B at the boundary point $\mathbf{c}$ for $\nu' = 1$.

Let $0 \leqslant \nu' < 1$ be arbitrarily chosen. Since $(\mathbf{m}, \mathbf{v}) \subseteq$ relint(B) and $0 < \psi < 1$, it follows that the point $\mathbf{v} + \psi(\mathbf{m}-\mathbf{v}) \in$ relint(B). Since $\mathbf{c} \in$ relbd(B) it follows from Theorem 2.10.10 that $(\mathbf{c}, \mathbf{v} + \psi(\mathbf{m}-\mathbf{v})) \subseteq$ relint(B). Observe that

$$\begin{aligned}
&\mathbf{k} + \nu(\mathbf{k}-\mathbf{z}) + \left(1 + \frac{(1-\psi)(1-\nu')}{\nu}\right)\Big(\mathbf{m} + \nu'(\mathbf{c}-\mathbf{m}) - \big(\mathbf{k} + \nu(\mathbf{k}-\mathbf{z})\big)\Big)\\
&= \mathbf{k} + \nu(\mathbf{k}-\mathbf{z}) + \left(1 + \frac{(1-\psi)(1-\nu')}{\nu}\right)\big((1-\vartheta)(\mathbf{m}-\mathbf{n}) + \nu'(\mathbf{c}-\mathbf{m}) + \nu(\mathbf{z}-\mathbf{k})\big)\\
&= \mathbf{m} + \left(\frac{(1-\psi)(1-\nu')(1-\vartheta)}{\nu}\right)(\mathbf{m}-\mathbf{n}) + (1-\psi)(1-\nu')(\mathbf{z}-\mathbf{k}) + \nu'(1-\psi)(\mathbf{v}-\mathbf{m})\\
&\qquad + \nu'\kappa(\mathbf{m}-\mathbf{n})\\
&= \mathbf{m} + (1-\psi)(\mathbf{v}-\mathbf{m}) + \left(\frac{(1-\psi)(1-\nu')(1-\vartheta)}{\nu} + \nu'\kappa\right)(\mathbf{m}-\mathbf{n})\\
&= \mathbf{c} + (1-\nu')\left(\kappa - \frac{(1-\psi)(1-\nu')(1-\vartheta)}{\nu}\right)(\mathbf{m}-\mathbf{n})\\
&= \mathbf{c} + (1-\nu')\left(1 - \frac{(1-\psi)(1-\nu')(1-\vartheta)}{\kappa\nu}\right)(\mathbf{v} + \psi(\mathbf{m}-\mathbf{v}) - \mathbf{c})\\
&= \left(1 - (1-\nu')\left(1 - \frac{(1-\psi)(1-\nu')(1-\vartheta)}{\kappa\nu}\right)\right)\mathbf{c}\\
&\qquad + (1-\nu')\left(1 - \frac{(1-\psi)(1-\nu')(1-\vartheta)}{\kappa\nu}\right)(\mathbf{v} + \psi(\mathbf{m}-\mathbf{v})).
\end{aligned}$$

To see that the point, above, on the ray through $\mathbf{k}+\nu(\mathbf{k}-\mathbf{z})$ with direction $\mathbf{m}+\nu'(\mathbf{c}-\mathbf{m}) - \big(\mathbf{k}+\nu(\mathbf{k}-\mathbf{z})\big)$ belongs to the open line segment $(\mathbf{c}, \mathbf{v}+\psi(\mathbf{m}-\mathbf{v})) \subseteq$ relint(B), verify that

$$0 < (1-\nu')\left(1 - \frac{(1-\psi)(1-\nu')(1-\vartheta)}{\kappa\nu}\right) < 1.$$

Since $\kappa > 0$ and $\nu > \dfrac{(1-\vartheta)(1-\psi)}{\kappa}$, it follows that $\kappa\nu > (1-\vartheta)(1-\psi)$. It follows from

$0 < \psi < 1$ and $0 \leqslant \vartheta < 1$ that

$$0 < 1 - \psi < 1 \qquad \text{and} \qquad 0 < 1 - \vartheta \leqslant 1. \tag{4.36}$$

Since $0 \leqslant \nu' < 1$, it follows that $0 < 1 - \nu \leqslant 1$. This together with (4.36) and Corollary A.2 implies that

$$\kappa\nu > (1-\psi)(1-\vartheta) > (1-\psi)(1-\nu')(1-\vartheta) > 0. \tag{4.37}$$

This means that

$$\kappa\nu - (1-\psi)(1-\nu')(1-\vartheta) > 0. \tag{4.38}$$

Notice that

$$\kappa\nu\left(1 - \frac{(1-\psi)(1-\nu')(1-\vartheta)}{\kappa\nu}\right) = \kappa\nu - (1-\psi)(1-\nu')(1-\vartheta) > 0.$$

In particular, this implies that $1 - \dfrac{(1-\psi)(1-\nu')(1-\vartheta)}{\kappa\nu} > 0$, since $\kappa\nu > 0$, by (4.37). Therefore,

$$(1-\nu')\left(1 - \frac{(1-\psi)(1-\nu')(1-\vartheta)}{\kappa\nu}\right) > 0.$$

Also, notice that

$$\kappa\nu\left(1 - \left(1 - \frac{(1-\psi)(1-\nu')(1-\vartheta)}{\kappa\nu}\right)\right) = (1-\psi)(1-\nu')(1-\vartheta) > 0.$$

This means that

$$\left(1 - \left(1 - \frac{(1-\psi)(1-\nu')(1-\vartheta)}{\kappa\nu}\right)\right) > 0,$$

since $\kappa\nu > 0$. Re-arrange the above inequality to get that $1 - \dfrac{(1-\psi)(1-\nu')(1-\vartheta)}{\kappa\nu} < 1$. This together with Corollary A.2 implies that

$$(1-\nu')\left(1 - \frac{(1-\psi)(1-\nu')(1-\vartheta)}{\kappa\nu}\right) < (1-\nu') \leqslant 1.$$

Hence, the ray through the point $\mathbf{k} + \nu(\mathbf{k} - \mathbf{z})$ with direction $\mathbf{m} + \nu'(\mathbf{c} - \mathbf{m}) - \big(\mathbf{k} + \nu(\mathbf{k} - \mathbf{z})\big)$ intersects relint(B) for $0 \leqslant \nu' < 1$.

Let $\nu' = 1$.

Sub-case (a): Suppose that $1 > \dfrac{(1-\vartheta)(1-\psi)}{\kappa}$.

Then, let $\mathbf{p}_2 = \mathbf{k} + (\mathbf{k} - \mathbf{z})$ and let $\mathbf{p}_1 = \mathbf{z}$. Choose $\mathbf{d}_1 = (\mathbf{z} - \mathbf{k}) + (\mathbf{k} - \mathbf{k}^+)$ and $\mathbf{d}_2 = (\mathbf{z} - \mathbf{k}) + (\mathbf{k} - \mathbf{k}^-)$. Notice that $\mathbf{p}_2 \notin B$ and $\mathbf{k} = \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)$.

Let $\mathbf{x} \in W_{[\mathbf{n},\mathbf{m})}$ be arbitrarily chosen. It follows that $\Pr(\mathbf{x}) \in [\mathbf{n}, \mathbf{m})$ and therefore, there exists $0 < \mho \leqslant 1$ such that $\Pr(\mathbf{x}) = \mho\mathbf{n} + (1-\mho)\mathbf{m}$. If $\mathbf{x} \in \left(W_{[\mathbf{n},\mathbf{m})}\right)_+ \setminus [\mathbf{n}, \mathbf{m})$, then there exists $0 < \Omega \leqslant 1$ such that $\mathbf{x} - \Pr(\mathbf{x}) = \Omega(\mathbf{k}^+ - \mathbf{k})$. This means that $\mathbf{k}^+ - \mathbf{k} = \dfrac{1}{\Omega}(\mathbf{x} - \Pr(\mathbf{x}))$ where $\dfrac{1}{\Omega} \geqslant 1$. The ray passing through $\mathbf{x}$ with direction $\mathbf{d}_1$ contains the point

$$\begin{aligned}
\mathbf{x} + \Omega(\mathbf{d}_1) &= \mathbf{x} + \Omega\left[(\mathbf{z} - \mathbf{k}) + \left(\mathbf{k} - \mathbf{k}^+\right)\right] \\
&= \mathbf{x} + \Omega(\mathbf{v} - \mathbf{m}) - (\mathbf{x} - \Pr(\mathbf{x})) \\
&= \mho\mathbf{n} + (1-\mho)\mathbf{m} + \Omega(\mathbf{v} - \mathbf{m}).
\end{aligned}$$

The points $\mathbf{v} + (1-\Omega)(\mathbf{m} - \mathbf{v}) \in (\mathbf{m}, \mathbf{v})$ and $\mathbf{u} + (1-\Omega)(\mathbf{n} - \mathbf{u}) \in (\mathbf{u}, \mathbf{n})$ belong to relint(B) since $(\mathbf{m}, \mathbf{v}), (\mathbf{u}, \mathbf{n}) \subseteq$ relint(B) and $0 < 1 - \Omega < 1$. It follows from Corollary 2.10.11 that $\big(\mathbf{v} + (1-\Omega)(\mathbf{m} - \mathbf{v}), \mathbf{u} + (1-\Omega)(\mathbf{n} - \mathbf{u})\big) \subseteq$ relint(B) and observe that it contains the point

$$\begin{aligned}
&(1-\mho)\big(\mathbf{v} + (1-\Omega)(\mathbf{m} - \mathbf{v})\big) + \mho\big(\mathbf{u} + (1-\Omega)(\mathbf{n} - \mathbf{u})\big) \\
&= \mathbf{v} + \mho(\mathbf{u} - \mathbf{v}) + (1-\Omega)(\mathbf{m} - \mathbf{v}) - \mho(1-\Omega)(\mathbf{m} - \mathbf{v}) + \mho(1-\Omega)(\mathbf{n} - \mathbf{u}) \\
&= \mathbf{m} + (\mathbf{v} - \mathbf{m}) + (1-\Omega)(\mathbf{m} - \mathbf{v}) + \mho(\mathbf{n} - \mathbf{m}) \\
&= \mho\mathbf{n} + (1-\mho)\mathbf{m} + \Omega(\mathbf{v} - \mathbf{m}).
\end{aligned}$$

Thus, $\mathbf{x} + \Omega(\mathbf{d}_1) \in$ relint$(B) \subseteq$ int(K). This means that the direction $\mathbf{d}_1$ illuminates $\mathbf{x}$.

If $\mathbf{x} \in \left(W_{[\mathbf{n},\mathbf{m})}\right)_- \setminus [\mathbf{n}, \mathbf{m})$, then a nearly identical proof can be used to show that the direction $\mathbf{d}_2$ illuminates the point $\mathbf{x}$.

If $\mathbf{x} \in [\mathbf{n}, \mathbf{m})$, then $\mathbf{x} = \Pr(\mathbf{x}) = \mho\mathbf{n} + (1-\mho)\mathbf{m}$, for some $0 < \mho \leqslant 1$. By convexity, $\mathbf{x}$ must be a cliff point. Let

$$\left\{\mathbf{x}^-\right\} = \{\mathbf{x} - \lambda\mathbf{e}_3 \mid \lambda \in \mathbb{R}\} \cap \left(\left[\mathbf{k}^-, \mathbf{m}\right] \cup \left[\mathbf{n}^-, \mathbf{k}^-\right]\right).$$

Since K is affine plane symmetric, $\mathbf{k}^+ - \mathbf{k} = \mathbf{k} - \mathbf{k}^-$. So, there exists $0 < \Omega' \leqslant 1$ such that $\mathbf{x} - \mathbf{x}^- = \Omega'(\mathbf{x}^+ - \mathbf{k})$. This means that $\mathbf{k}^+ - \mathbf{k} = \frac{1}{\Omega'}(\mathbf{x} - \mathbf{x}')$ where $\frac{1}{\Omega'} \geqslant 1$. Observe that

$$\begin{aligned}
\mathbf{x} + \frac{1-\psi}{1+\frac{1-\psi}{\Omega'}}\mathbf{d}_1 &= \mathbf{x} + \frac{1-\psi}{1+\frac{1-\psi}{\Omega'}}\Big((\mathbf{z}-\mathbf{k}) + \big(\mathbf{k}-\mathbf{k}^+\big) \Big) \\
&= \mathbf{x} + \frac{1-\psi}{1+\frac{1-\psi}{\Omega'}}(\mathbf{v}-\mathbf{m}) - \frac{1}{\Omega'}\left(\frac{1-\psi}{1+\frac{1-\psi}{\Omega'}}\right)(\mathbf{x}-\mathbf{x}') \\
&= \left(1 - \frac{1}{\Omega'}\left(\frac{1-\psi}{1+\frac{1-\psi}{\Omega'}}\right)\right)\mathbf{x} + \frac{1-\psi}{1+\frac{1-\psi}{\Omega'}}(\mathbf{v}-\mathbf{m}) + \frac{1}{\Omega'}\left(\frac{1-\psi}{1+\frac{1-\psi}{\Omega'}}\right)\mathbf{x}^- \\
&= \left(1 - \frac{1}{\Omega'}\left(\frac{1-\psi}{1+\frac{1-\psi}{\Omega'}}\right)\right)\big(\mathbf{x} + (1-\psi)(\mathbf{v}-\mathbf{m})\big) + \frac{1}{\Omega'}\left(\frac{1-\psi}{1+\frac{1-\psi}{\Omega'}}\right)\mathbf{x}^- \\
&= \left(1 - \frac{1}{\Omega'}\left(\frac{1-\psi}{1+\frac{1-\psi}{\Omega'}}\right)\right)\Big((1-\mho)\big(\mathbf{v} + \psi(\mathbf{m}-\mathbf{v})\big) + \mho\big(\mathbf{u} + \psi(\mathbf{n}-\mathbf{u})\big)\Big) \\
&\quad + \frac{1}{\Omega'}\left(\frac{1-\psi}{1+\frac{1-\psi}{\Omega'}}\right)\mathbf{x}^-.
\end{aligned}$$

Sub-case (b): Suppose that $1 \leqslant \frac{(1-\vartheta)(1-\psi)}{\kappa}$.

Then, choose some $\nu > \frac{(1-\vartheta)(1-\psi)}{\kappa}$. Let $\mathbf{p}_2$ be the point $\mathbf{k}+\nu(\mathbf{k}-\mathbf{z})$ on the line through the points $\mathbf{k}$ and $\mathbf{z}$. Also, let $\mathbf{p}_1$ be the point $\mathbf{z}+(1-\nu)(\mathbf{k}-\mathbf{z})$ on the line through the points $\mathbf{k}$ and $\mathbf{z}$. Choose $\mathbf{d}_1 = (\mathbf{p}_1 - \mathbf{k}) + (\mathbf{k} - \mathbf{k}^+) = \nu(\mathbf{z}-\mathbf{k}) + (\mathbf{k}-\mathbf{k}^+)$ and $\mathbf{d}_2 = (\mathbf{p}_1 - \mathbf{k}) + (\mathbf{k}-\mathbf{k}^-) = \nu(\mathbf{z}-\mathbf{k}) + (\mathbf{k}-\mathbf{k}^-)$. Notice that ${}^1\!/_2(\mathbf{p}_1+\mathbf{p}_2) = {}^1\!/_2\big(\mathbf{z} + (1-\nu)(\mathbf{k}-\mathbf{z}) + \mathbf{k} + \nu(\mathbf{k}-\mathbf{z})\big) = \mathbf{k}$.

Chapter 5

Conclusion

B.V. Dekster proved that illumination conjecture holds for 3-dimensional convex bodies with affine plane symmetry [21]. His proof cleverly combined theory from elementary geometry with non-trivial results from convex analysis. This thesis re-examined his work and used it as a prototype for a more detailed proof of some cases, while at the same time correcting some minor flaws in the original proof.

Bibliography

[1] *Fundamental Concepts in Modern Analysis*. World Scientific, 1999.

[2] A.D. Aleksandrov. *Convex Polyhedra*. Springer, 2006.

[3] P.S. Aleksandrov. *Combinatorial Topology: Vol.: 1*. Graylock Press, 1956.

[4] T.M. Apostol. *Mathematical Analysis*. Addison-Wesley, second edition, 1977.

[5] T.M. Apostol. *Linear Algebra: A First Course with Applications to Differential Equations*. Wiley, 1997.

[6] Katie Barich. Proving completeness of the hausdorff induced metric space, 2011. Senior Project.

[7] A. Barvinok. *A Course in Convexity*. Graduate Studies in Mathematics, Volume 54. American Mathematical Society, 2002.

[8] R.V. Benson. *Euclidean geometry and convexity*. McGraw-Hill, 1966.

[9] L.D. Berkovitz. *Convexity and Optimization in R^n*. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2003.

[10] Károly Bezdek. *Classical topics in discrete geometry*. Springer, 2010.

[11] Károly Bezdek. The problem of illumination of the boundary of a convex body by affine subspaces. *Mathematika*, 38:362–375, 1991.

[12] Károly Bezdek. On the illumination of smooth convex bodies. *Archiv der Mathematik*, 58(6):611–614, 1992.

[13] V. Boltyansi and I. Gohberg. Stories about covering and illuminating of convex bodies. *Nieuw Archief voor Wiskundee. Vierde Serie*, 13(1):1–26, 1995.

[14] V. Boltyanski, H. Martini, and P.S. Soltan. *Excursions into Combinatorial Geometry*. Universitext. Springer Berlin Heidelberg, 2012.

[15] V.G. Boltyanski. The problem of illuminating the boundary of a convex body. *Izv. Mold. Filiala Akad. Nauk SSSR*, 10(76):77–84, 1960.

[16] V.G. Boltyanskii. Solution of the illumination problem for three dimensional convex bodies. *Doklady Akademii Nauk*, 375:298–301, 2000.

[17] D.A. Brannan, M.F. Esplen, and J.J. Gray. *Geometry*. Cambridge University Press, 2011.

[18] P. Brass, W.O.J. Moser, and J. Pach. *Research Problems in Discrete Geometry*. Springer, 2006.

[19] V.V. Buldygin and A.B. Kharazishvili. *Geometric Aspects of Probability Theory and Mathematical Statistics*. Springer, 2000.

[20] J. Cooper. *Working Analysis*. Elsevier Academic Press, 2005.

[21] Boris Dekster. Each convex body in $\mathbb{E}^3$ symmetric about a plane can be illuminated by 8 directions. *Journal of Geometry*, 69(1):37–50, 2000.

[22] S. Epp. *Discrete Mathematics with Applications*. Cengage Learning, third edition, 2003.

[23] J.E. Gentle. *Matrix Algebra: Theory, Computations, and Applications in Statistics*. Springer, 2007.

[24] M.S. Gockenbach. *Finite-Dimensional Linear Algebra*. Discrete Mathematics and Its Applications. CRC Press, 2011.

[25] I. Ts. Gokhberg and A. S. Markus. On covering convex figures by figures similar to them. *Izv. Mold. Filiala Akad. Nauk SSSR*, 76(10):87–90, 1960.

[26] P.M. Gruber. *Convex and Discrete Geometry*. Grundlehren der mathematischen Wissenschaften. Springer, 2007.

[27] H. Hadwiger. Ungelöste Probleme. *Elemente der Mathematik*, 15(6):130–131, 1960.

[28] K. Hoffman. *Analysis in Euclidean Space*. Prentice-Hall Inc., 1975.

[29] R. B. Howlett. Metric spaces: Lecture 4. `http://www.maths.usyd.edu.au/u/bobh/UoS/MATH3901/00met04.pdf`, 2000.

[30] Asaf Karagila. Cantor intersection theorem in $\mathbb{R}^n$. Mathematics Stack Exchange, 2014. `http://math.stackexchange.com/q/761581`.

[31] J.H. Kwak and S. Hong. *Linear Algebra*. Birkhäuser, 2013.

[32] Marek Lassak. Solution of hadwiger's covering problem for centrally symmetric convex bodies in $\mathbb{E}^3$. *Journal of the London Mathematical Society*, 30(3):501–511, 1984.

[33] P.D. Lax. *Linear Algebra and Its Applications*. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2013.

[34] S.R. Lay. *Analysis: with an introduction to proof*. Pearson Prentice Hall, 2005.

[35] S.R. Lay. *Convex Sets and Their Applications*. Dover Publications, 2007.

[36] F.W. Levi. Überdeckung eines eibereiches durch parallelverschiebung seines offenen kerns. *Archiv der Mathematik*, 6(5):369–370, 1955.

[37] T.W. Ma. *Classical Analysis on Normed Spaces*. World Scientific, 1995.

[38] J.M. Maruskin. *Essential Linear Algebra*. Solar Crest Publishing, 2012.

[39] P. McMullen and G.C. Shephard. *Convex Polytopes and the Upper Bound Conjecture*. London Mathematical Society Lecture Note Series 3. Cambridge University Press, 1971.

[40] C.D. Meyer. *Matrix Analysis and Applied Linear Algebra:*. Society for Industrial and Applied Mathematics, 2000.

[41] Jesper Michael Møller and Morten Poulsen. Munkres §17. `http://www.math.ku.dk/~moller/e02/3gt/opg/S17.pdf`, 2004.

[42] J.R. Munkres. *Topology*. Prentice Hall, 2000.

[43] W.K. Nicholson. *Linear Algebra with Applications*. McGraw-Hill Ryerson, fifth edition, 2006.

[44] J. O'Rourke. *Art Gallery Theorems and Algorithms*. Oxford University Press, 1987.

[45] Ioannis Papadoperakis. An estimate for the problem of illumination of the boundary of a convex body in E^3. *Geometriae Dedicata*, 75(3):275–285, 1999.

[46] I.R. Porteous. *Topological Geometry*. Cambridge University Press, 1981.

[47] V.V. Prasolov and S. Ivanov. *Problems and Theorems in Linear Algebra*. American Mathematical Society, 1994.

[48] Jing-Hui Qiu. Mazur's theorem and boundary points of convex bodies. *Soochow Journal of Mathematics*, 31(1):131–141, 2005.

[49] J. Ratcliffe. *Foundations of Hyperbolic Manifolds*. Graduate Texts in Mathematics. Springer, 2006.

[50] W. Rudin. *Principles of Mathematical Analysis*. International series in pure and applied mathematics. McGraw-Hill, 1976.

[51] Rolf Schneider. *Convex Bodies: The Brunn-Minkowski Theory*. Cambridge University Press, second expanded edition, 2013.

[52] Reinhard Schultz. Affine transformations and convexity. `http://math.ucr.edu/~res/math145A-2014/affine+convex.pdf`, 2014.

[53] Reinhard Schultz. Iii. spaces with special properties. `http://math.ucr.edu/~res/math205A-2014/math205Asolutions03.pdf`, 2014.

[54] R.K. Sharma. *Algebra I: A Basic Course in Abstract Algebra*. Pearson, 2011.

[55] P. Soberón. *Problem-Solving Methods in Combinatorics: An Approach to Olympiad Problems*. Springer, 2013.

[56] R.R. Stoll. *Linear Algebra and Matrix Theory*. International Series in Pure and Applied mathematics. McGraw-Hill, 1952.

[57] E. Suhubi. *Functional Analysis*. Springer, 2013.

[58] L. Szabó. Recent results on illumination problems. pages 207–221, 1997.

[59] J.B.H. Urruty and C. Lemaréchal. *Fundamentals of Convex Analysis*. Grundlehren Text Editions Series. Springer, 2001.

[60] Roger Webster. *Convexity*. Oxford University Press, 1994.

Appendix A

Elementary Number Theory

Proposition A.1. *Let λ and μ be real numbers such that $1 \geqslant \lambda \geqslant 0$ and $\mu \geqslant 0$. Then, $\lambda\mu \leqslant \mu$.*

Proof. Suppose $0 \leqslant \lambda \leqslant 1$. It follows that $1 - \lambda \geqslant 0$. Also, suppose $\mu \geqslant 0$. The product of two non-negative numbers is non-negative. Therefore, it follows that $\mu(1-\lambda) \geqslant 0$. Re-arrange this inequality to get $\lambda\mu \leqslant \mu$. ∎

Corollary A.2. *Let λ and μ be real numbers such that $0 \leqslant \lambda, \mu \leqslant 1$. Then, $\lambda\mu \leqslant \mu$ and $\lambda\mu \leqslant \lambda$.*

Proposition A.3. *Let λ_1, λ_2 and λ_3 be real numbers such that $\lambda_3 \geqslant \lambda_1$ and $\lambda_2, \lambda_3 > 0$. Then,*

$$\frac{\lambda_1 + \lambda_2}{\lambda_3 + \lambda_2} \geqslant \frac{\lambda_1}{\lambda_3}.$$

Proof. Suppose $\lambda_3 \geqslant \lambda_1$ and $\lambda_3 > 0$. Then, $\lambda_3 - \lambda_1 \geqslant 0$. Recall that the product of two non-negative numbers is a non-negative number. Therefore, $\lambda_3^2 - \lambda_3\lambda_1 = \lambda_3(\lambda_3 - \lambda_1) \geqslant 0$. Observe that

$$\frac{1}{\lambda_3}\left(\lambda_3^2 - \lambda_3\lambda_1\right) = \frac{\lambda_3^2}{\lambda_3} - \frac{\lambda_3\lambda_1}{\lambda_3} = \lambda_3 - \lambda_1 \geqslant 0.$$

Also, recall that a non-negative number can either be written as the product of two non-negative numbers or the product of two non-positive numbers. It follows that $\frac{1}{\lambda_3} \geqslant 0$.

Furthermore, suppose $\lambda_2 > 0$. It follows that $\lambda_3\lambda_2 - \lambda_1\lambda_2 = \lambda_2(\lambda_3 - \lambda_1) \geqslant 0$ and $\lambda_3 + \lambda_2 > 0$. Also, it follows that $\lambda_3 + \lambda_2 \geqslant \lambda_1 + \lambda_2$. This inequality can be re-arranged as $(\lambda_3 + \lambda_2) - (\lambda_1 + \lambda_2) \geqslant 0$. Use the property of the product of non-negative numbers mentioned above to get $(\lambda_3 + \lambda_2)^2 - (\lambda_3 + \lambda_2)(\lambda_1 + \lambda_2) = (\lambda_3 + \lambda_2)((\lambda_3 + \lambda_2) - (\lambda_1 + \lambda_2)) \geqslant 0$. Observe that

$$\frac{1}{\lambda_3 + \lambda_2}\left((\lambda_3 + \lambda_2)^2 - (\lambda_3 + \lambda_2)(\lambda_1 + \lambda_2)\right) = (\lambda_3 + \lambda_2) - (\lambda_1 + \lambda_2) \geqslant 0.$$

It follows that $\dfrac{1}{\lambda_3 + \lambda_2} \geqslant 0$. Thus,

$$\frac{1}{\lambda_3\left(\lambda_3 + \lambda_2\right)} = \frac{1}{\lambda_3} \cdot \frac{1}{\lambda_3 + \lambda_2} \geqslant 0.$$

Hence,

$$\frac{\lambda_1 + \lambda_2}{\lambda_3 + \lambda_2} - \frac{\lambda_1}{\lambda_3} = \frac{\lambda_3\left(\lambda_1 + \lambda_2\right) - \lambda_1\left(\lambda_3 + \lambda_2\right)}{\lambda_3\left(\lambda_3 + \lambda_2\right)} = \left(\lambda_3 \lambda_2\right) \cdot \frac{1}{\lambda_3\left(\lambda_3 + \lambda_2\right)} \geqslant 0.$$

Re-arrange this inequality to get

$$\frac{\lambda_1 + \lambda_2}{\lambda_3 + \lambda_2} \geqslant \frac{\lambda_1}{\lambda_3}.$$

■